Final test | MATH 3280 3.00 | December 6, 2019

Given name and surname:______________________________

Student No:______________________________

Signature:______________________________

INSTRUCTIONS:

1. Please write everything in **ink**.
2. This exam is a 'closed book' test, duration **180** minutes.
3. Only non-programmable calculators are permitted.
4. There are fourteen questions and a bonus question.

USEFUL FORMULAS:

For $x \geq 0,\ t \in [0,\ 1)$ and $k = 0,\ 1,\ 2,\ \ldots$, if the uniform distribution of deaths assumption holds for the life-status (x), then the following is true

$$_{t+k}p_x \approx (1-t)\,{}_kp_x + t\,{}_{k+1}p_x.$$

Let $C : [0,\ 1]^2 \to [0,\ 1]$ denote a copula function, and let $u, v \in [0,\ 1]$, then the Fréchet-Hoeffding bounds state

$$\max(u+v-1, 0) \leq C(u,v) \leq \min(u,v).$$

GOOD LUCK!

1. Recall that we denote by (u) a general life-status. Choose $(u) = (\overset{1}{x} : \overline{n}\rceil),\ x \geq 0,\ n = 1, 2, \ldots$

   - What insurance contract does this life-status correspond to? Explain in **one** sentence.
   - Write formally the random variable $T(u) = T(\overset{1}{x} : \overline{n}\rceil)$ as well as the probability $\mathbb{P}\left(T(\overset{1}{x} : \overline{n}\rceil) \geq t\right),\ t \geq 0$.
   - Assume that the life-status (x) admits the uniform distribution of deaths approximation, check whether the life-status $(\overset{1}{x} : \overline{n}\rceil)$ also admits the uniform distribution of deaths approximation.

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2. A life of age 50 is subject to an extra hazard during the year of age 50 to 51. If the standard probability of death from age 50 to age 51 is 0.06, and if the extra risk may be expressed by an addition to the standard force of mortality that decreases uniformly from 0.03 at the beginning of the year to 0 at the end of the year, calculate the probability that the life will survive to age 51.

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Cont.

3. Let $F_{T(x)}(s) = s$ and $F_{T(y)}(t) = t$, where $x, y \geq 0$ and $s, t \in [0, 1]$. Also, let the copula function that describes the dependence between the random variables $T(x)$ and $T(y)$ be given by

$$C(s,t) = \frac{1}{\alpha} \log \left(1 + \frac{(\exp\{\alpha F_{T(x)}(s)\} - 1)(\exp\{\alpha F_{T(y)}(t)\} - 1)}{\exp\{\alpha\} - 1}. \right), \ \alpha \neq 0,$$

for $s, t \in [0, 1]$.

- Derive, in terms of the copula function above, the cumulative distribution function of the random variable $T(x : y)$.
- For $\tilde{T}(x : y) = T(x)^a \vee T(y)^b, \ a, b > 0$, find the cumulative distribution function of the random variable $\tilde{T}(x : y)$.

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Cont.

4. Recall that the random variable Z is said to be distributed Weibull with the shape parameter $\gamma > 0$ and rate parameter $\lambda > 0$, succinctly, $Z \sim Wei(\gamma,\ \lambda)$, if it has the following cumulative distribution function

$$\mathbb{P}(Z \leq z) = 1 - \exp\{-\lambda z^{\gamma}\},\ z \geq 0.$$

- Check what assumptions must be made on the parameters' choice, so that the following statement is true: *If the random variables $T(x)$ and $T(y)$ are independent and both distributed Weibull, so is the random variable $T(x : y)$.*
- Let $T^*(x) \sim Wei(\gamma,\ \lambda_x^*)$, $T^*(y) \sim Wei(\gamma,\ \lambda_y^*)$ and $Z \sim Wei(\gamma,\ \lambda)$, all mutually independent. Set

$$T(x) = \min\left(\mathrm{T^*(x), Z}\right) \text{ and } \mathrm{T(y)} = \min\left(\mathrm{T^*(y), Z}\right).$$

  Find $\mathbb{P}(T(x) > u,\ T(y) > v),\ u, v \in [0,\ \infty)$.
- Use the Fréchet-Hoeffding bounds to derive the bounds for the probability ${}_tp_{x:y}$ in terms of ${}_tp_x$ and ${}_tp_y$, where $x, y, t \geq 0$.

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Cont.

5. Let $Q \in Mat_{m\times m}$ denote transition probability matrix in a homogeneous multi-state model with $m \in \mathbb{N}$ states. Prove that ${}_kQ = Q^k,\ k \in \mathbb{N}$. Also show that ${}_kQ$ is a stochastic matrix in the sense that all the elements in its each row sum up to one.

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Cont.

6. Order the following in terms of magnitude (and explain):

$$q_x^{'(i)},\ q_x^{(i)},\ m_x^{'(i)},$$

where $x \geq 0$ and $i \in \mathbb{N}$. Recall

$$m_x^{'(i)} = \frac{\int_0^1 {}_tp_x^{'(i)} \mu_x^{(i)}(t)\ \mathrm{d}t}{\int_0^1 {}_tp_x^{'(i)} \mathrm{d}t}$$

is the central rate of mortality.

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7. On the basis of a triple decrement table, display an expression for the probability that (20) will not terminate before age (65) for cause 2.

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Cont.

8. Let, for $t \in [0,\ 1]$ and $i \in \mathbb{N}$,

$$w^{(\tau)}(t) = \frac{{}_tp_x^\tau}{\int_0^1 {}_tp_x^\tau \mathrm{d}t}$$

and

$$w^{(i)}(t) = \frac{{}_tp_x^{\prime(i)}}{\int_0^1 {}_tp_x^{\prime(i)} \mathrm{d}t}.$$

Assume that $i \in \mathbb{N}$ and at least one other cause of decrement have positive forces of decrement on the interval $[0,\ 1]$. Show that $w^{(\tau)}(0) > w^{(i)}(0)$.

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9. For a double-decrement table, you are given that $m_{40}^{(\tau)} = 0.2$ and $q_{40}^{\prime(1)} = 0.1$. Compute $q_{40}^{\prime(1)}$ assuming the uniform distribution of decrements in the associated single decrement tables.

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10. Recall that the Gompertz's law of mortality is formally given by $\mu_x(t) = bc^x,\ b, c > 0$. Find $w \geq 0$, such that

$$ {}_tp_w = {}_tp_{x:y}, $$

for $t \geq 0$ and when the future life-time random variables $T(x)$ and $T(y)$ are independent.

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Cont.

11. Formulate the future life-time random variables $T(x:y)$ and $T(\overline{x:y})$ in terms of the random variables $T(x)$ and $T(y)$. Given that the random variables $T(x)$ and $T(y)$ are uncorrelated, show that the random variables $T(x:y)$ and $T(\overline{x:y})$ are correlated.

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12. Check the following bound under the assumption of independence and dependence

$${}_tp_{\overset{1}{x}:y} \leq {}_tp_{x:\overset{2}{y}},$$

for $t \geq 0$.

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Cont.

13. Prove or disprove the identity

$$ {}_tp_{x:y}\mu_{x:y}(t) + {}_tp_{\overline{x:y}}\mu_{\overline{x:y}}(t) = {}_tp_x\mu_x(t) + {}_tp_y\mu_y(t), $$

where $t,\ x,\ y \geq 0$.

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14. Consider a pair of individuals of the same age, whose future life-time random variables are independent copies of the random variable $T(x),\ x \geq 0$. Evaluate the integral

$$\int_0^n {}_tp_{x:x}\mu_{x:x}(t)\mathrm{d}t,$$

where $n \in \mathbb{R}_+$.

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Cont.

15. Bonus Find $\overset{\circ}{e}_{x:y}$ if $q_x = q_y = 1$ and the deaths are uniformly distributed over the year of age for each one of (x) and (y), where $x, y \geq 0$.

The End.