Definition 0.1

- Cartesian product of two closed intervals $\mathcal{B} := [x_1,\ x_2] \times [y_1,\ y_2]$.
- The vertices of the rectangular $\mathcal{B}$ are the points $(x_1, y_1),\ (x_1, y_2),\ (x_2, y_1),\ (x_2, y_2)$.
- The unit product square $\mathcal{I}^2 := [0,\ 1] \times [0,\ 1]$.
- Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two non-empty subsets of $\overline{\mathbf{R}}$, and let $H : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{R} \subset \mathbf{R}$, then the H-volume of $\mathcal{B}$ is

$$V_H(\mathcal{B}) = H(x_2, y_2) - H(x_1, y_2) - H(x_2, y_1) + H(x_2, y_2).$$

- The function H is 2-increasing if $V_H(\mathcal{B}) \geq 0$ for all rectangles $\mathcal{B}$ that lie in the domain of H.

Note.

The fact that a bivariate function is non-decreasing in two arguments does not mean it is 2-increasing. Check at home what happens with $H(x, y) = \max(x, y)$, i.e., find $V_H(\mathcal{I}^2)$.

Definition 0.2

A copula C is a function from $\mathcal{I}^2$ to $\mathcal{I}$, such that:

- For every u, v in $\mathcal{I}$,

$$C(u,0) = C(0,v) = 0$$

  and

$$C(u,1) = u \text{ and } C(1,v) = v.$$

- C is 2-increasing, i.e., for $u_1 \leq u_2$ and $v_1 \leq v_2$, and u_1, u_2, v_1, v_2 in $\mathcal{I}$,we have that

$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0.$$

Note.

It can be easily seen that $C(u,v) = V_C([0,u] \times [0,v])$. Indeed,

$$V_C([0,u]\times[0,v]) = C(u,v) - C(0,v) - C(u,0) + C(0,0) = C(u,v).$$

Proposition 0.1

Frechet-Hoeffding bounds Let C be a copula, then for every (u, v) in its domain, we have that

$$\max(u + v - 1, 0) \le C(u, v) \le \min(u, v).$$

Proof.

Let (u, v) be an arbitrary pair in $\mathcal{I}^2$. Clearly $C(u, v) \le C(u, 1) = u$ and also $C(u, v) \le C(1, v) = v$. Thus $C(u, v) \le \min(u, v)$. Furthermore, $V_C([u, 1] \times [v, 1]) \ge 0$ implies

$$\begin{aligned} V_C([u, 1] \times [v, 1]) &= C(1, 1) - C(u, 1) - C(1, v) + C(u, v) \\ &= 1 - u - v + C(u, v) \ge 0, \end{aligned}$$

thus $C(u, v) \ge u + v - 1$. Then taking into account that $C(u, v) \ge 0$, completes the proof. □

Proposition 0.2 (Sklar's theorem)

Let H be a joint c.d.f. with margins F and G. Then there exists a copula C, such that for all x, y in $\overline{\mathbf{R}}$, it holds that

$$H(x, y) = C(F(x), G(y)).$$

If F and G are continuous, then C is unique. The converse is true as well.

Proposition 0.3

Let X and Y be two continuous r.v.'s with a copula $C_{X,Y}$. If $a(\cdot)$ and $b(\cdot)$ are strictly increasing on the ranges of X and Y, respectively, then $C_{a(X),b(Y)} = C_{X,Y}$.

Note.

Copulas as measures of dependence are invariant under strictly increasing transformations.

Proof.

Let $F_1,\ G_1,\ F_2,\ G_2$ be the c.d.f.'s of $X,\ Y,\ a(X)$ and $b(Y)$. We have that

$$F_2(x) = \mathbf{P}[a(X) \le x] = \mathbf{P}[X \le a^{-1}(x)] = F_1(a^{-1}(x)).$$

Also

$$G_2(y) = G_1(b^{-1}(y)).$$

Thus, for any x and y in $\overline{\mathbf{R}}$, we have that, using Sklar's theorem,

$$\begin{aligned} C_{a(X),b(Y)}(F_2(x), G_2(y)) &= \mathbf{P}[a(X) \le x, b(Y) \le y] \\ &= \mathbf{P}[X \le a^{-1}(x), Y \le b^{-1}(y)] \\ &= C_{X,Y}(F_1(a^{-1}(x)), G_1(b^{-1}(y))) \\ &= C_{X,Y}(F_2(x), G_2(y)), \end{aligned}$$

which completes the proof. □

Definition 0.3

The copula

$$C_\theta(u, v) = uv + \theta uv(1 - u)(1 - v),$$

with u and v in $\mathcal{I}$ is called the Farlie- Gumbel - Morgenstern copula. Here $\theta \in [-1,\ 1]$.

Note.

If $\theta = 0$, then the copula becomes $C_0 = uv$, that is the independent copula.