We seek ${}_3Q_0^{(2,2)}$, which is the $(2,2)$-entry of $\boldsymbol{Q}^3$. Rather than proceeding as in the preceding Example, consider the following approach. Note that if $\boldsymbol{e}_j$ denotes an $n \times 1$ column matrix with 1 as its j^{th} entry and 0 as its other entries, then for any $k \times n$ matrix $\boldsymbol{M}$ the product $\boldsymbol{M}\boldsymbol{e}_j$ is just the j^{th} column of $\boldsymbol{M}$. Therefore the desired ${}_3Q_0^{(2,2)}$, which is the $(2,2)$-entry of $\boldsymbol{Q}^3$, is just the bottom entry of

$$\begin{aligned}\boldsymbol{Q}^3\boldsymbol{e}_2 = \boldsymbol{Q}^2(\boldsymbol{Q}\boldsymbol{e}_2) &= \begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix}\begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix}\begin{bmatrix}0.4\\0.7\end{bmatrix}\\ &= \begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix}\begin{bmatrix}0.52\\0.61\end{bmatrix} = \begin{bmatrix}0.556\\0.583\end{bmatrix},\end{aligned}$$

giving 0.583 for the answer.

There's another probability that will prove central to computations in Section 2.2: for a subject in State $\#s$ at time n, the probability of making the transition from State $\#i$ at time $n+k$ to State $\#j$ at time $n+k+1$. In order to possibly make this transition, the subject first must be in State $\#i$ at time $n+k$. Since the subject is now in State $\#s$ at time n, the probability of this is ${}_kQ_n^{(s,i)}$. The probability of the transition then from State $\#i$ to State $\#j$ is $Q_{n+k}^{(i,j)}$. The product ${}_kQ_n^{(s,i)}Q_{n+k}^{(i,j)}$ of these two probabilities gives the probability of the transition in question. That is,

(1.23) **Theorem** (future transition probabilities). Given that a subject is in State $\#s$ at time n, the probability of making the transition from State $\#i$ at time $n+k$ to State $\#j$ at time $n+k+1$ is given by ${}_kQ_n^{(s,i)}Q_{n+k}^{(i,j)}$.

Problems

1. A basic aggregate survival model as in Example 1.1 follows the DeMoivre Law with ultimate age $\omega = 100$. As in Example 1.8, find the matrix $\boldsymbol{Q}_{30}$ for a person aged $x = 60$.
   [Answer: the first row contains 0.9 and 0.1, the second 0 and 1.]
2. Consider a multiple-life model as in Example 1.10 for independent lives aged $x = 60$ and $y = 75$ subject to a DeMoivre Law with $\omega = 100$. As in Example 1.10, find $Q_{10}^{(1,2)}$.
   [Answer: $\frac{29}{450}$.]
3. For the model in Example 1.17, find ${}_3Q^{(2,1)}$.
   [Answer: 0.608.]
4. As in Example 1.5, consider a driver-ratings model in which drivers move among the classifications Preferred, Standard, and Substandard at the end of each

year. Each year: 60% of Preferreds are reclassified as Preferred, 30% as Standard, and 10% as substandard; 50% of Standards are reclassified as Standard, 30% as Preferred, and 20% as Substandard; and 60% of Substandards are reclassified as Substandard, 40% as Standard, and 0% as Preferred. Find the probability that a driver, classified as Standard at the start of the first year, will be classified as Standard at the start of the fourth year.
[Answer: 0.409.]

5. Consider the situation in Problem 4 again. Find the probability that a driver, classified as Standard at the start of the first year, will be classified as Standard at the start of each of the first four years.
[Answer: 0.125.]

6. Consider the CCRC model in Example 1.21. Find the probability that a resident, in Independent Living at time 1, will not be Gone at time 3.
[Answer: 0.8175.]

7. Consider a disability model with four states, numbered in order: Active, Temporarily Disabled, Permanently Disabled, and Inactive. Suppose that the transition-probability matrices for a new employee (at time 0) are as given in the Illustrative Matrices in Section 3.1. For an Active employee at time 1, find the probability the employee is Inactive at time 4.
[Answer: 0.3535.]

8. Consider a four-state non-homogeneous Markov Chain with transition probability matrices given by the Illustrative Matrices in Section 3.1. For a subject in State #2 at time 3, find the probability that the subject transitions from State #1 at time 5 to State #3 at time 6.
[Answer; 0.033.]

9. (Theory.) Extend Example 1.17 in general for homogeneous Markov Chains with two states to prove that ${}_2\boldsymbol{Q} = \boldsymbol{Q}^2$.

10. (Theory.) Extend Problem 9 to non-homogeneous Markov Chains with r states to prove that ${}_2\boldsymbol{Q}_n = \boldsymbol{Q}_n \boldsymbol{Q}_{n+1}$.

11. (Theory.) Extend Problem 10 to prove Theorem 1.18 on longer-term probabilities.

3

An illustrative non-homogeneous Markov Chain

This chapter presents a set of illustrative transition-probability matrices and cash flows for use in examples and problems.

3.1 Illustrative transition-probability matrices

Consider a non-homogeneous Markov Chain with four states numbered 1, 2, 3, 4. The transition-probability matrices given below have 0 or 1 in certain positions so that the models make sense for disability models as in Example 1.11 and Continuing Care Retirement Community models as in Example 1.13. The other probabilities have been chosen arbitrarily and of course are unlikely to be appropriate for real-life situations.

Transition-probability matrices are given at times 0, 1, 2, 3, 4, 5, 6, and 7, with the same matrix for all times $n \geq 8$; this final matrix is chosen so that the subject is certain to reach State #4 by time 9 and then remain there forever.

$$\boldsymbol{Q}_0 = \begin{bmatrix} 0.80 & 0.10 & 0.05 & 0.05 \\ 0.20 & 0.60 & 0.10 & 0.10 \\ 0 & 0 & 0.80 & 0.20 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{Q}_1 = \begin{bmatrix} 0.70 & 0.15 & 0.10 & 0.05 \\ 0.20 & 0.50 & 0.20 & 0.10 \\ 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} 0.60 & 0.15 & 0.15 & 0.10 \\ 0.20 & 0.40 & 0.25 & 0.15 \\ 0 & 0 & 0.60 & 0.40 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{Q}_3 = \begin{bmatrix} 0.50 & 0.20 & 0.20 & 0.10 \\ 0.20 & 0.30 & 0.35 & 0.15 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{Q}_4 = \begin{bmatrix} 0.40 & 0.20 & 0.20 & 0.20 \\ 0.10 & 0.30 & 0.30 & 0.30 \\ 0 & 0 & 0.40 & 0.60 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{Q}_5 = \begin{bmatrix} 0.30 & 0.20 & 0.30 & 0.20 \\ 0.10 & 0.20 & 0.40 & 0.30 \\ 0 & 0 & 0.30 & 0.70 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{Q}_6 = \begin{bmatrix} 0.20 & 0.20 & 0.30 & 0.30 \\ 0.10 & 0.10 & 0.40 & 0.40 \\ 0 & 0 & 0.20 & 0.80 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{Q}_7 = \begin{bmatrix} 0.10 & 0.10 & 0.30 & 0.50 \\ 0.05 & 0.05 & 0.30 & 0.60 \\ 0 & 0 & 0.10 & 0.90 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{and, for } n \geq 8, \quad \boldsymbol{Q}_n = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.2 Illustrative cash flows upon transitions

This section presents some illustrative cash flows upon transitions between states in the non-homogeneous Markov Chain described in Section 3.1. The particular values for the cash flows are not intended to be meaningful—rather they were chosen to be easily distinguishable from one another so that you can see from where values come in Examples.

For convenience in displaying the values, I've entered the cash flow ${}_{\ell+1}C^{(i,j)}$ that occurs at time $\ell + 1$ as the (i,j)-entry of a matrix ${}_{\ell+1}\boldsymbol{C}$.

$${}_1\boldsymbol{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad {}_2\boldsymbol{C} = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \\ 0 & 0 & 19 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$${}_3\boldsymbol{C} = \begin{bmatrix} 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 \\ 0 & 0 & 29 & 30 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad {}_4\boldsymbol{C} = \begin{bmatrix} 31 & 32 & 33 & 34 \\ 35 & 36 & 37 & 38 \\ 0 & 0 & 39 & 40 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$${}_5\boldsymbol{C} = \begin{bmatrix} 41 & 42 & 43 & 44 \\ 45 & 46 & 47 & 48 \\ 0 & 0 & 49 & 50 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad {}_6\boldsymbol{C} = \begin{bmatrix} 51 & 52 & 53 & 54 \\ 55 & 56 & 57 & 58 \\ 0 & 0 & 59 & 60 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$${}_7\boldsymbol{C} = \begin{bmatrix} 61 & 62 & 63 & 64 \\ 65 & 66 & 67 & 68 \\ 0 & 0 & 69 & 70 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad {}_8\boldsymbol{C} = \begin{bmatrix} 71 & 72 & 73 & 74 \\ 75 & 76 & 77 & 78 \\ 0 & 0 & 79 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\text{and, for } \ell \geq 8, \quad {}_{\ell+1}\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 81 \\ 0 & 0 & 0 & 82 \\ 0 & 0 & 0 & 83 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$