Mathematics of Life Contingencies. Math 3280 3.00 F

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Homework 8

Unless otherwise indicated, all lives in the following questions are subject to the same law of mortality and their times until death are independent random variables.

1. For two lives with lifetime variables S and T, you are given that f(s, t) = (s+t)/125 for $0 < s,\ t < 5$.

   Calculate $p_{2:2}$.

   solution

$$p_{2:2} = \frac{s(3:3)}{s(2:2)}$$

   The function s(x,x) is calculated by double integrating the density function from x to 5.

$$\begin{aligned} s(x,x) &= \int_x^5 \int_x^5 \frac{s+t}{125} dsdt \\ &= \frac{(5-x)^2(5+x)}{125} \end{aligned}$$

   We conclude:

$$\begin{aligned} s(2,2) &= \frac{3^2(7)}{125} = 0.504 \\ s(3,3) &= \frac{2^2(8)}{125} = 0.256 \\ p_{2:2} &= \frac{s(3,3)}{s(2,2)} = \frac{0.256}{0.504} = 0.507937 \end{aligned}$$

2. Three lives have independent future lifetimes. The forces of mortality are $\mu_x(t) = 0.02$ for the first life, $\mu_y(t) = 0.03$ for the second life, and $\mu_z(t) = 0.04$ for the third life.

Calculate the probability that the first death from the three lives will occur at least five years from the current date, but no later than ten years from the current date.

solution

The joint status $\mu_{xyz}(t)$ for the joint status of the three lives is 0.02+0.03+0.04=0.09. Therefore,

$$_5p_{xyz} = e^{-5(0.09)} = 0.637628$$

$$_{10}p_{xyz} = e^{-10(0.09)} = 0.406570$$

$$_5p_{xyz} -_{10} p_{xyz} = 0.637628 - 0.406570 = 0.231058$$

3. For two lives with future lifetimes S and T, you are given

$$f(s,t) = \frac{s^2+t}{540}, \quad 0 < s, \ t \leq 6$$

Calculate the probability that the joint status survives at least three years.

solution

In order for the joint status to survive at least three years, both S and T must be greater than 3. Thus we integrate both variables from 3 to the upper limit of 6.

$$1/540 \int_3^6 \int_3^6 (s^2+t)dsdt = \frac{1}{540}\int_3^6 (63+3t)dt$$

$$= \frac{1}{540}(189 + 3(\frac{6^2-3^2}{2})) = 0.425$$

4. Let T be the random variable for the future lifetime of the joint status (x, y) where (x) is female and (y) is male. (x) and (y) are independent.

You are given:

1) $\mu_x^F = (2+x)^{-1}$ where μ_x^F is the force of mortality for females.

2) $\mu_y^M = a(2+y)^{-1}$ where μ_y^M is the force of mortality for males.

3) $F_{T(3,2)}(20) = 179/180$.

Calculate a.

solution

The survival probabilities are:

$$ {}_{20}p_3^F = exp(-\int_0^{20} \mu_{3+t}^F dt) $$

$$ = exp(-\int_0^{20} (2 + (3+t))^{-1} dt) $$

$$ = exp(-ln(5+t))|_0^{20} = 1/5 $$

$$ {}_{20}p_2^M = exp(-\int_0^{20} \mu_{2+t}^M dt) $$

$$ = exp(-a\int_0^{20} (2 + (2+t))^{-1} dt) $$

$$ = exp(-aln(4+t))|_0^{20} = (\frac{1}{6})^a $$

You are given that the complement of the survival function of the joint life status, F(20)=179/180. So

$$ (\frac{1}{5})(\frac{1}{6})^a = \frac{1}{180} $$

$$ 6^a = 36 $$

$$ a = 2 $$

5. For two lives (x) and (y) with independent future lifetimes, the force of mortality for each is 0.01.

   Calculate $\mu_{\overline{xy}}(10)$.

   solution

For each life, ${}_tp_x = e^{-0.01t}$. Therefore

$$ {}_tp_{xy} = e^{-0.02t} $$

$$ {}_{10}p_{\overline{xy}} = {}_{10}p_x + {}_{10}p_y - {}_{10}p_{xy} $$

$$ = 2e^{-0.1} - e^{-0.2} = 0.990944 $$

Then $\mu_{\overline{xy}}(10)$ is the density of (y) dying if (x) died plus the desity of (x) dying if(y) died over ${}_{10}p_{\overline{xy}}$. Both are the same, so we double one.

$$ {}_{10}q_x \ {}_{10}p_x \ \mu_x(10) = (1 - e^{-0.1})(e^{-0.1})(0.01) = 0.000861067 $$

$$ \mu_{\overline{xy}}(10) = \frac{2(0.000861067)}{0.990944} = 0.001738 $$

6. You are given:

   1) Mortality follows De Moivre's law with w=105.

   2) (45) and (65) have independent future lifetimes.

   Calculate $\mathring{e}_{\overline{45:65}}$.

   solution

   $\mathring{e}_{45} = 30$, $\mathring{e}_{65} = 20$.

   And also ${}_tp_{45:65} = (\frac{60-t}{60})(\frac{40-t}{40})$ for $0 < t < 40$.

   $\mathring{e}_{45:65} = \int_0^{40} (\frac{60-t}{60})(\frac{40-t}{40}) dt = \frac{140}{9}$

   $\mathring{e}_{\overline{45:65}} = 30 + 20 - \frac{140}{9} = 34\frac{4}{9}$.

7. For a partnership for two partners, you are given:

   1) The partnership is dissolved if either partner quits.

   2) The probability that one of the partners will quit by time t is t/20 for $t < 20$.

   3) The probability that the other partner will quit by time t is t/10 for $t < 10$.

4) The amounts of time until a partner quits are independent.

5) The partnership will be dissolved at the end of five years if not dissolved sooner due to a partner quitting.

Calculate the expected lifetime of the partnership.

solution

We need the 5-year temporary future lifetime of the partnership. The formula is

$$\mathring{e}_{0:0} = \int_0^5 {}_tp_{0:0}dt = \int_0^5 (\frac{20-t}{20})(\frac{(10-t)}{10}dt$$

$$= \frac{1}{200}\int_0^5 (10+10-t)(10-t)dt$$

$$= 0.005(5(10^2-5^2)+\frac{10^3-5^3}{3}) = 3\frac{1}{3}$$

8. Mortality for two lives follows De Moivre's law with w=105. Future lifetimes for the two lives are independent.

   Calculate $\mathbf{Cov}(T(35:30), T(\overline{35:30}))$.

   solution

   Expand $\mathbf{Cov}(T(xy), T(\overline{xy}))$

$$\mathbf{Cov}(T(xy), T(\overline{xy})) = E[T(xy)T(\overline{xy})] - E[T(xy)]E[T(\overline{xy})]$$

$$= E[T(xy)T(\overline{xy})] - \mathring{e}_{xy}\ \mathring{e}_{\overline{xy}}$$

$$= E[T(xy)T(\overline{xy})] - \mathring{e}_{xy}\ (\mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy})$$

   Further more,

$$E[T(xy)T(\overline{xy})] = E[T(x)T(y)]$$

$$= \mathbf{Cov}(T(x), T(y)) + \mathring{e}_x\mathring{e}_y$$

Combine the expression,

$$\mathbf{Cov}(T(xy), T(\overline{xy})) = \mathbf{Cov}(T(x), T(y)) + \mathring{e}_x\mathring{e}_y - \mathring{e}_{xy}\mathring{e}_x - \mathring{e}_{xy}\mathring{e}_y + \mathring{e}_{xy}^2$$

$$= \mathbf{Cov}(T(x), T(y)) + (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy})$$

In this question, we have

$$\mathring{e}_{35} = 35$$

$$\mathring{e}_{30} = 37.5$$

And

$$\mathring{e}_{35:30} = 24.111$$

$$\mathbf{Cov}(T(35:30), T(\overline{35:30})) = (\mathring{e}_{35} - \mathring{e}_{35:30})(\mathring{e}_{30} - \mathring{e}_{35:50}) = 145.790$$

GOOD LUCK!