Quiz 6 MATH 3280 3.00 Nov 111, 2020

Given name and surname:____________________________________

Student No:____________________________________

Signature:____________________________________

INSTRUCTIONS:

1. Please write everything in **ink**.
2. This quiz is a 'closed book' test, duration **15** minutes.
3. Only non-programmable calculators are permitted.
4. The text has two pages, and it contains two questions. Read the question carefully. Fill in answers in designated spaces. Your work must justify the answer you give. Answers without supporting work will **not** be given credit.

GOOD LUCK!

Question 1 Prove that for two RVs $T(x) \geq 0,\ T(y) \geq 0$, we have

$$\mathrm{Cov}(T(x), T(y)) = \int_0^\infty \int_0^\infty \left(\mathbb{P}\left(T(x) \geq u, T(y) \geq v\right) - \mathbb{P}\left(T(x) \geq u\right)\mathbb{P}\left(T(y) \geq v\right)\right) \mathrm{d}u\mathrm{d}v,$$

given that the covariance is well-defined and finite. Use this identity to show that if the RVs $T(x) \geq 0,\ T(y) \geq 0$ are PQD, that is positively quadrant dependent, then these RVs are positively correlated.

Question 2 Let $T'(x),\ T'(y),\ \xi$ be independent RVs distributed gamma with shape parameters $\gamma'_x,\ \gamma'_y,\ \gamma$, respectively, and rate parameter λ; all parameters are positive. Also, set $T(x) = T'(x) + \xi$ and $T(y) = T'(y) + \xi$. Show that the RVs $T(x)$ and $T(y)$ are PQD. Derive the covariance of the RVs $T(x)$ and $T(y)$.

Q.1

If well-defined and finite, the covariance of the RVs $T(x), T(y)$

$$\mathrm{Cov}\left(T(x), T(y)\right) = \mathbb{E}\left[T(x)T(y)\right] - \mathbb{E}\left[T(x)\right]\mathbb{E}\left[T(y)\right],$$

where

$$\mathbb{E}\left[T(x)T(y)\right] = \mathbb{E}\left[\int_0^{T(x)}\int_0^{T(y)} ds\,dt\right] = \mathbb{E}\left[\int_0^\infty\int_0^\infty \mathbb{I}\{T(x) \geq t\}\,\mathbb{I}\{T(y) \geq s\}\,ds\,dt\right]$$

$$= \mathbb{E}\left[\int_0^\infty\int_0^\infty \mathbb{I}\{T(x) \geq t, T(y) \geq s\}\,ds\,dt\right] = \int_0^\infty\int_0^\infty \mathbb{E}\left[\mathbb{I}\{T(x) \geq t, T(y) \geq s\}\right] ds\,dt$$

$$= \int_0^\infty\int_0^\infty \mathbb{P}\left(T(x) \geq t, T(y) \geq s\right) ds\,dt.$$

and in a similar fashion, e.g.,

$$\mathbb{E}\left[T(x)\right] = \int_0^\infty \mathbb{P}\left(T(x) \geq t\right) dt.$$

Hence the desired formula follows.

Q.2

$$\mathrm{Cov}\left(T(x), T(y)\right) = \mathrm{Cov}\left(T'(x) + \xi, T'(y) + \xi\right) \overset{\perp\!\!\!\perp}{=} \mathrm{Var}(\xi) = \frac{\gamma}{\lambda^2}.$$

Also note that $\mathbb{P}(\xi \geq s, \xi \geq t) = \mathbb{P}(\xi \geq \min(s,t)) \geq \mathbb{P}(\xi \geq s)\mathbb{P}(\xi \geq t)$, $s,t \geq 0$ and so (ξ, ξ) is PQD. Similarly, $(\xi + a, \xi + b)$ is also PQD.

The End.

Question 1 Prove that for two RVs $T(x) \geq 0,\ T(y) \geq 0$, we have

$$\operatorname{Cov}(T(x), T(y)) = \int_0^\infty \int_0^\infty \left(\mathbb{P}\left(T(x) \geq u, T(y) \geq v\right) - \mathbb{P}\left(T(x) \geq u\right)\mathbb{P}\left(T(y) \geq v\right)\right) \mathrm{d}u \mathrm{d}v,$$

given that the covariance is well-defined and finite. Use this identity to show that if the RVs $T(x) \geq 0,\ T(y) \geq 0$ are PQD, that is positively quadrant dependent, then these RVs are positively correlated.

Question 2 Let $T'(x),\ T'(y),\ \xi$ be independent RVs distributed gamma with shape parameters $\gamma'_x,\ \gamma'_y,\ \gamma$, respectively, and rate parameter λ; all parameters are positive. Also, set $T(x) = T'(x) + \xi$ and $T(y) = T'(y) + \xi$. Show that the RVs $T(x)$ and $T(y)$ are PQD. Derive the covariance of the RVs $T(x)$ and $T(y)$.

As $f(u) = \mathbb{I}\{u \geq s\}$ for $s \geq 0$ is non-decreasing, then RVs $\mathbb{I}\{\xi + a \geq s\}$ & $\mathbb{I}\{\xi + b \geq t\}$ are PQD and so

$$\mathbb{E}\left[\mathbb{I}\{\xi + a \geq s\}\,\mathbb{I}\{\xi + b \geq t\}\right] \geq \mathbb{E}\left[\mathbb{I}\{\xi + a \geq s\}\right]\mathbb{E}\left[\mathbb{I}\{\xi + b \geq t\}\right]$$

or because of $\perp\!\!\!\perp$

$$\mathbb{E}\left[\mathbb{I}\{\xi + T'(x) \geq s\} \Big| \mathbb{I}\{\xi + T'(y) \geq t\} \Big| T'(x) = a, T'(y) = b\right]$$

$\geq$

$$\mathbb{E}\left[\mathbb{I}\{\xi + T'(x) \geq s\} \Big| T'(x) = a\right]\mathbb{E}\left[\mathbb{I}\{\xi + T'(y) \geq t\} \Big| T'(y) = b\right]$$

or, as a, b can be any ≥ 0,

$$\mathbb{E}\left[\mathbb{I}\{\xi + T'(x) \geq s\} \Big| \mathbb{I}\{\xi + T'(y) \geq t\} \Big| T'(x), T'(y)\right]$$

$\geq$

$$\mathbb{E}\left[\mathbb{I}\{\xi + T'(x) \geq s\} \Big| T'(x)\right] \cdot \mathbb{E}\left[\mathbb{I}\{\xi + T'(y) \geq t\} \Big| T'(y)\right]$$

hence by conditioning and the towering property of expectations

$$\mathbb{E}\left[\mathbb{I}\{\xi + T'(x) \geq s\} \Big| \mathbb{I}\{\xi + T'(y) \geq t\}\right] \geq \mathbb{E}\left[\mathbb{I}\{\xi + T'(x) \geq s\}\right] \cdot \mathbb{E}\left[\mathbb{I}\{\xi + T'(y) \geq t\}\right].$$

The End.