Mid-Term test MATH 3280 3.00 October 30, 2020

Given name and surname: ____________________

Student No: ____________________

Signature: ____________________

INSTRUCTIONS:

1. Please write everything in **ink**.
2. This exam is a 'closed book' test, duration **60** minutes.
3. Only non-programmable calculators are permitted.
4. There are five questions (each 20%) and a bonus question (10%).

USEFUL FORMULAS:

For $x \geq 0,\ t \in [0,\ 1)$ and $k = 0,\ 1,\ 2,\ \ldots$, if the uniform distribution of deaths assumption holds for the life-status (x), then the following is true

$$_{t+k}p_x \approx (1-t)_k p_x + t_{k+1}p_x.$$

GOOD LUCK!

1. You are given the following excerpt from a life table (the select period is two years)

x	$100q_{[x]}$	$100q_{[x]+1}$	$100q_{x+2}$
30	0.438	0.574	0.699
31	0.453	0.599	0.734
32	0.472	0.634	0.790
33	0.510	0.680	0.856
34	0.551	0.737	0.937

Compute the following deferred probability ${}_{1|}q_{[30]+1}$.

Sol

$${}_{1|}q_{[30]+1} = p_{[30]+1} \cdot q_{[30]+2}$$
$$= \left(1 - q_{[30]+1}\right) \cdot q_{32}$$
$$= \left(1 - \frac{1}{100}\, 0.574\right) \frac{1}{100}\, 0.699$$
$$= 0.00695$$

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2. Let the future life-time random variables $T(x)$ and $T(y)$ be independent; $x, y \geq 0$. You are given that $q_x = 0.080,\ q_y = 0.004$ and that ${}_tp_x = 1 - t^2 q_x$ as well as ${}_tp_y = 1 - t^2 q_y$, both for $0 \leq t \leq 1$. Find the probability density function of the random variable $T(x:y)$.

Sol

We have, for $t \in [0,1]$,

$${}_tp_{x:y} \overset{\perp}{=} {}_tp_x \, {}_tp_y = (1 - t^2\, 0.08)(1 - t^2\, 0.004)$$

$$= 1 - 0.084\, t^2 + 0.00032\, t^4.$$

Then

$${}_tp_{x:y}\, \mu_{x:y}(t) = -\frac{d}{dt}\, {}_tp_{x:y} = 2 \cdot 0.084\, t - 4\, 0.00032\, t^3$$

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3. Let the random variables $T(x)$ and $T(y)$ be independent, each distributed exponentially with mean 20. Find the value of $Var(T(x:y))$ as well as of $Cov(T(x:y), T(\overline{x:y}))$, where $x, y \geq 0$.

Sol

$\overline{{}_tp_{x:y}} = e^{-0.05t - 0.05t} = e^{-0.1t}$, for $t \geq 0$.

Hence $T(x:y) \sim Exp(0.1)$, and $Var(T(x:y)) = \frac{1}{(0.1)^2} = 100$

Further

$Cov(T(x:y), T(\overline{x:y})) = \mathbb{E}[T(x:y)\, T(\overline{x:y})] - \mathbb{E}[T(x:y)]\, \mathbb{E}[T(\overline{x:y})]$

$= \mathbb{E}[T(x)\, T(y)] - \mathring{e}_{x:y}\, \mathring{e}_{\overline{x:y}}$

$\overset{\perp}{=} \mathring{e}_x \mathring{e}_y - \mathring{e}_{x:y}\, \mathring{e}_{\overline{x:y}}$

$= \mathring{e}_x \mathring{e}_y - \mathring{e}_{x:y} (\mathring{e}_x + \mathring{e}_y - \mathring{e}_{x:y})$

$= 20 \cdot 20 - 10 (20 + 20 - 10)$

$= 400 - 300 = 100$

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4. Let $T'(x)$, $T'(y)$, ξ be independent random variables distributed exponentially with the respective rate parameters $\lambda'_x > 0$, $\lambda'_y > 0$, $\lambda > 0$. Set $T(x) = (T'(x) \wedge \xi)$ and $T(y) = (T'(y) \wedge \xi)$. (Recall that, for instance, $\mathbb{P}(\xi > t) = e^{-(\lambda t)}$, $\lambda > 0$, $t \geq 0$.) Derive the probability that (x) dies before (y).

Sol

$$\mathbb{P}(T(x) < T(y)) = \int_0^\infty \int_u^\infty f_{T(x),T(y)}(u,v)\, dv\, du$$

$$= \int_0^\infty \int_u^\infty \exp\{-\lambda_x u - (\lambda + \lambda_y) v\} \lambda_x (\lambda + \lambda_y)\, dv\, du$$

$$= \int_0^\infty \exp\{-\lambda_x u\} \lambda_x \left(\int_u^\infty \exp\{-(\lambda + \lambda_y) v\} (\lambda + \lambda_y)\, dv \right) du$$

$$= \int_0^\infty \exp\{-\lambda_x u\} \lambda_x \exp\{-(\lambda + \lambda_y) u\}\, du$$

$$= \int_0^\infty \exp\{-(\lambda_+ u)\} \lambda_x = \frac{\lambda_x}{\lambda_+},$$

$$\lambda_+ = \lambda + \lambda_x + \lambda_y.$$

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5. You are given the following table

x	e_x
75	10.5
76	10.0
77	9.5

Compute the probability that a life aged 75 exactly survives to the age 77 exactly.

Sol

As $e_x = \mathbb{E}[K(x)] = \mathbb{E}[K(x)\, \mathbb{I}\{K(x) \geq 1\}] + \mathbb{E}[K(x)\, \mathbb{I}\{K(x) < 1\}]$

$= p_x\, \mathbb{E}[K(x) \mid K(x) \geq 1] + q_x\, \mathbb{E}[K(x) \mid K(x) < 1]$

$= p_x \left(\mathbb{E}[K(x) - 1 \mid K(x) \geq 1] + 1 \right)$

$= p_x (e_{x+1} + 1)$

So $p_x = e_x / (e_{x+1} + 1)$

And ${}_2p_{75} = p_{75}\, p_{76} = \dfrac{e_{75}}{e_{76} + 1} \cdot \dfrac{e_{76}}{e_{77} + 1} = 0.909$

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6. You are given the following life table

x	l_x
50	99,813
51	97,702
52	95,046

Compute ${}_{0.75}p_{50.5}$ using the UDD and CFM approximation methods.

Sol

$p_{50} = 97,702/99,813 = 0.978850$

$p_{51} = 95,046/97,702 = 0.972815$

Also: ${}_{0.5}p_{50} \cdot {}_{0.75}p_{50.5} = p_{50} \cdot {}_{0.25}p_{51}$, hence

$${}_{0.75}p_{50.5} = \frac{p_{50} \cdot {}_{0.25}p_{51}}{{}_{0.5}p_{50}}$$

$$\overset{UDD}{=} \frac{p_{50}\,(1-0.25(1-p_{51}))}{1-0.5\,(1-p_{50})} = 0.982588$$

For CFM

$\mu_{50} = -\ln(p_{50}) = 0.021377$

$\mu_{51} = -\ln(p_{51}) = 0.027561$

So ${}_{0.75}p_{50.5} = {}_{0.5}p_{50.5} \cdot {}_{0.25}p_{51} \overset{CFM}{=} {}_{0.5}p_{50} \cdot {}_{0.25}p_{51}$

$= e^{-0.5 \cdot 0.021377} \cdot e^{-0.25 \cdot 0.027561} = 0.982574$