Mid-Term test MATH 3280 3.00 October 30, 2020

Given name and surname:________________________

Student No:________________________

Signature:________________________

INSTRUCTIONS:

1. Please write everything in **ink**.
2. This exam is a 'closed book' test, duration **60** minutes.
3. Only non-programmable calculators are permitted.
4. There are five questions (each 20%) and a bonus question (10%).

USEFUL FORMULAS:

For $x \geq 0,\ t \in [0,\ 1)$ and $k = 0,\ 1,\ 2,\ \ldots$, if the uniform distribution of deaths assumption holds for the life-status (x), then the following is true

$$_{t+k}p_x \approx (1-t)\,{}_kp_x + t\,{}_{k+1}p_x.$$

GOOD LUCK!

1. You are given the following life table

x	l_x	d_x	p_x	q_x
95	–	–	–	0.4
96	–	–	0.2	–
97	–	72	–	1

Also, you are given that $l_{90} = 1000$ and $l_{93} = 825$. Under the assumption that deaths are uniformly distributed over each year of age, compute the probability that (90) dies between ages 93 and 95.5.

Sol

We look for $_{3|2.5}q_{90} = \frac{l_{90+3} - l_{90+3+2.5}}{l_{90}} = \frac{l_{93} - l_{95.5}}{l_{90}} = \frac{l_{93} - (l_{95} - 0.5 d_{95})}{l_{90}}$

$= (*)$

Note that $l_{97} = \frac{d_{97}}{q_{97}} = 72$

$l_{96} = \frac{l_{97}}{p_{96}} = 360$

$l_{95} = \frac{l_{96}}{p_{95}} = 600$

$d_{95} = l_{95} - l_{96} = 240$

$(*) = \frac{825 - (600 - 0.5 \cdot 240)}{1{,}000} = 0.3450.$

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Cont.

2. Assume that the future life-times of the life-statuses (x) and (y) are independent and follow Makeham's law, that is $\mu(t) = c + a \times b^t$, $t > 0$, where a, b, c are positive parameters, and $x, y \geq 0$ are not necessarily equal. Find a joint life-status $(w : w)$ that has force of mortality equal to the force of mortality of the joint life-status $(x : y)$ for all $t > 0$.

Sol

$\mu_{w:w}(t) = \mu_{x:y}(t)$

So $2(c + ab^t b^w) = 2c + ab^t(b^x + b^y)$

Or $2c + 2ab^t b^w = 2c + ab^t(b^x + b^y)$

or $b^w = \frac{1}{2}(b^x + b^y)$

or $w \ln b = \ln \frac{1}{2}(b^x + b^y)$

Hence $w = \ln 0.5(b^x + b^y) / \ln b$.

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Cont.

3. Use the illustrative life-table and compute the probability ${}_{2.75}q_{85.5}$ using the CFM approximation method.

Sol

We seek ${}_{2.75}q_{85.5} = 1 - {}_{2.75}p_{85.5} = 1 - {}_{0.5}p_{85.5} \cdot {}_{2}p_{86} \cdot {}_{0.25}p_{88}$

$\overset{CFM}{=} 1 - {}_{0.5}p_{85} \cdot {}_{2}p_{86} \cdot {}_{0.25}p_{88}$

$= 1 - (p_{85})^{0.5} \cdot p_{86} \, p_{87} \cdot (p_{88})^{0.25}$

$= 1 - \left(1 - \frac{1}{1,000}(123.8867)\right)^{0.5} \cdot \left(1 - \frac{1}{1,000} 134.9367\right) \cdot \left(1 - \frac{1}{1,000} 146.8926\right)$

$\cdot \left(1 - \frac{1}{1,000} 159.8121\right)^{0.25}$

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4. Let $T'(x),\ T'(y),\ \xi$ be independent random variables distributed exponentially with the respective rate parameters $\lambda'_x > 0,\ \lambda'_y > 0,\ \lambda > 0$. Set $T(x) = (T'(x) \wedge \xi)$ and $T(y) = (T'(y) \wedge \xi)$. (Recall that, for instance, $\mathbb{P}(\xi > t) = e^{-\lambda t},\ \lambda > 0,\ t \geq 0$.) Derive the probability $\mathbb{P}(T(x:y) \geq t),\ t \geq 0$. Also, show that the probability of simultaneous death is not in generally zero and derive an expression for this probability.

Sol

$${}_tp_{x:y} = \mathbb{P}(T'(x) \geq t,\ T'(y) \geq t,\ \xi \geq t) \overset{\perp\!\!\!\perp}{=} e^{-t(\lambda+\lambda_x+\lambda_y)},\quad t \geq 0$$

Also

$$\mathbb{P}(T(x) = T(y)) = \mathbb{P}(T'(x) > \xi,\ T'(y) > \xi) = \int_0^\infty \mathbb{P}(T'(x) > \xi,\ T'(y) > \xi \mid \xi = t)\, f_\xi(t)\,dt$$

$$\overset{\perp\!\!\!\perp}{=} \int_0^\infty e^{-\lambda_x t}\, e^{-\lambda_y t}\, \lambda\, e^{-\lambda t}\,dt$$

$$= \frac{\lambda}{\lambda+\lambda_x+\lambda_y} \neq 0$$

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5. The following excerpt from a life table applies to each of two independent lives (80) and (81)

x	q_x
80	0.50
81	0.75
82	1.00

Compute $q_{80:81}$ and $q_{\overline{80:81}}$.

Sol

$$q_{80:81} = 1 - p_{80:81} \overset{4}{=} 1 - p_{80}\, p_{81} = 1 - (1-q_{80})(1-q_{81})$$

$$= 1 - 0.5 \cdot 0.25 = 0.875$$

Also

$$q_{\overline{80:81}} = q_{80} \cdot q_{81} = 0.50 \cdot 0.75 = 0.325$$

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6. The function $x \mapsto l_x\mu_x$ is constant for all $0 \le x < \omega(>0)$. For $\omega = 100$, compute the variance of the random variable $T(88)$.

Sol

Since $l_x \mu_x$ is constant, l_x is linear. Hence the RV $T(88) \sim Uni[0,12]$ So

$$\mathbb{E}[T(88)] = \int_0^{12} t \frac{1}{12} dt = \frac{1}{12} \frac{t^2}{2}\Big|_0^{12} = 6$$

$$\mathbb{E}[T(88)^2] = \int_0^{12} t^2 \frac{1}{12} dt = \frac{1}{12} \frac{t^3}{3}\Big|_0^{12} = \frac{1}{12}\frac{12^3}{3} = \frac{144}{3}$$

$$Var(T(88)) = \frac{144}{3} - \frac{144}{4} = \frac{4 \cdot 144 - 3 \cdot 144}{12} = \frac{144}{12} = 12$$