MID-TERM TEST MATH 3280 3.00 OCTOBER 30, 2020

Given name and surname:________________________

Student No:________________________

Signature:________________________

INSTRUCTIONS:

1. Please write everything in **ink**.
2. This exam is a 'closed book' test, duration **60** minutes.
3. Only non-programmable calculators are permitted.
4. There are five questions (each 20%) and a bonus question (10%).

USEFUL FORMULAS:

For $x \geq 0,\ t \in [0,\ 1)$ and $k = 0,\ 1,\ 2,\ \ldots$, if the uniform distribution of deaths assumption holds for the life-status (x), then the following is true

$$_{t+k}p_x \approx (1-t)_k p_x + t_{k+1}p_x.$$

GOOD LUCK!

1. In a special mortality table with a select period of one year, the following relationships are true for all ages, $x \in \{0,\ 1,\ \ldots,\}$

$$
\begin{aligned}
{}_{0.5}q_{[x]} &= 0.25q_x \\
{}_{0.5}q_{[x]+0.5} &= 0.45q_x
\end{aligned}
$$

Express $p_{[x]}$ in terms of p_x. Find an expression for the full expectancy of life of the life-status $[x]$.

Sol

$$
\begin{aligned}
p_{[x]} &= {}_{0.5}p_{[x]} \cdot {}_{0.5}p_{[x]+0.5} = (1 - {}_{0.5}q_{[x]})(1 - {}_{0.5}q_{[x]+0.5}) \\
&= (1 - 0.25\,q_x)(1 - 0.45\,q_x) \\
&= (1 - 0.25(1 - p_x))(1 - 0.45(1 - p_x)) \\
&= (0.75 + 0.25\,p_x)(0.55 + 0.45\,p_x) \\
&= 0.4125 + 0.475\,p_x + 0.1125\,p_x^2.
\end{aligned}
$$

Also:

$$
e_{[x]} = \sum_{k=1}^{\infty} {}_k p_{[x]} = \sum_{k=1}^{\infty} \frac{l_{[x]+k}}{l_{[x]}} = \frac{l_x}{l_{[x]}} \sum_{k=1}^{\infty} \frac{l_{x+k}}{l_x} = \frac{l_x}{l_{[x]}} e_x
$$

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Cont.

2. Using the Illustrative Life Table, compute the constant force of mortality applicable to a life aged between ages 67 and 68 exact. Then compute ${}_{0.5}q_{67.25}$ using the constant force of mortality assumption.

Sol

We have

$$p_{67} = \exp\left\{-\int_{67}^{68} \mu_{67}\,dt\right\} \Rightarrow \mu_{67} = -\ln(p_{67}) = -\ln(1 - q_{67})$$

$$= -\ln\left(1 - \frac{1}{1000}\,25.4391\right) = 0.025768$$

Then

$${}_{0.5}q_{67.25} = 1 - {}_{0.5}p_{67.25} = 1 - {}_{0.5}p_{67}$$

$$= 1 - \exp\{-0.5\,\mu_{67}\} = 0.0128$$

.

Cont.

3. Evaluate

$$\frac{\partial}{\partial x}\overset{\circ}{e}_{x:y}$$

under the assumption of independence of the random variables $T(x)$ and $T(y)$, where $x, y > 0$.

$$\frac{\partial}{\partial x}\overset{\circ}{e}_{x:y} = \frac{\partial}{\partial x}\int_0^\infty {}_tp_x \, {}_tp_y \, dt = \int_0^\infty \frac{\partial}{\partial x} {}_tp_x \, {}_tp_y \, dt$$

$$= \int_0^\infty \frac{\partial}{\partial x} \exp\left\{-\int_0^t \mu(x+s)\,ds\right\} {}_tp_y \, dt$$

$$= -\int_0^\infty \exp\left\{-\int_0^t \mu(x+s)\,ds\right\}\left(\frac{\partial}{\partial x}\int_0^t \mu(x+s)\,ds\right) {}_tp_y \, dt$$

$$= -\int_0^\infty {}_tp_x \, {}_tp_y \left(\int_0^t \frac{\partial}{\partial x}\mu(x+s)\,ds\right) dt$$

$$= -\int_0^\infty {}_tp_x \, {}_tp_y \left(\int_0^t \frac{\partial}{\partial s}\mu(x+s)\,ds\right) dt$$

$$= -\int_0^\infty {}_tp_x \, {}_tp_y \left(\mu(x+t) - \mu(x)\right) dt$$

$$= \int_0^\infty {}_tp_x \, {}_tp_y \left(\mu(x) - \mu(x+t)\right) dt$$

$$= \mu(x)\,\overset{\circ}{e}_{x:y} - \int_0^\infty {}_tp_x \, {}_tp_y \, \mu(x+t)\,dt$$

.

Cont.

4. Let $T'(x),\ T'(y),\ \xi$ be independent random variables distributed Pareto with the respective shape parameters $\gamma'_x > 0,\ \gamma'_y > 0,\ \gamma > 0$ and the common scale parameter $\lambda > 0$. Set $T(x) = (T'(x) \wedge \xi)$ and $T(y) = (T'(y) \wedge \xi)$. (Recall that, for instance, $\mathbb{P}(\xi > t) = (1 + t/\lambda)^{-\gamma},\ \lambda > 0,\ \gamma > 0,\ t \geq 0$.) Derive the probability $\mathbb{P}(T(x : y) \geq t),\ t \geq 0$. Use this probability to find $\mu_{x:y}(t) = \mu((x : y) + t),\ t > 0$.

Sol

$$ {}_tp_{x:y} = \mathbb{P}(T(x:y) \geq t) = \mathbb{P}(T'(x) \wedge \xi \geq t,\ T'(y) \wedge \xi \geq t) $$

$$ = \mathbb{P}(T'(x) \geq t,\ T'(y) \geq t,\ \xi \geq t) $$

$$ \overset{\perp\!\!\!\perp}{=} \mathbb{P}(T'(x) \geq t)\,\mathbb{P}(T'(y) \geq t)\,\mathbb{P}(\xi \geq t) $$

$$ = \left(1 + \frac{t}{\lambda}\right)^{-\gamma'_x} \left(1 + \frac{t}{\lambda}\right)^{-\gamma'_y} \left(1 + \frac{t}{\lambda}\right)^{-\gamma} $$

$$ = \left(1 + \frac{t}{\lambda}\right)^{-(\gamma + \gamma'_x + \gamma'_y)},\ t \geq 0. $$

Also, we obtain

$$ -\frac{d}{dt}\,{}_tp_{x:y} = \frac{1}{\lambda}(\gamma + \gamma'_x + \gamma'_y)\left(1 + \frac{t}{\lambda}\right)^{-(\gamma + \gamma'_x + \gamma'_y + 1)} $$

and so

$$ \mu_{x:y}(t) = \frac{\gamma + \gamma'_x + \gamma'_y}{\lambda}\left(1 + \frac{t}{\lambda}\right)^{-1} = \frac{\gamma + \gamma'_x + \gamma'_y}{\lambda + t},\ t > 0. $$

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Cont.

5. Let the random variables $T(x)$ and $T(y)$ be independent and distributed $Uni[0,\ 1]$. Simplify the expression for the joint probability density function of the random variable $T(x:y)$ as much as possible.

Sol

$${}_tp_{x:y} \overset{ii}{=} {}_tp_x\ {}_tp_y = (1-t)(1-t) = 1-2t+t^2,\ t\in[0,1]$$

Hence

$$-\frac{d}{dt}\,{}_tp_{x:y} = -\frac{d}{dt}(1-2t+t^2) = 2-2t = 2(1-t),\quad t\in(0,1)$$

which is the PDF

Also

$$\mathbb{P}(T(x:y)\geq t \mid T(x)<1,\ T(y)<1)$$

$$= \frac{\mathbb{P}(T(x)\geq t,\ T(y)\geq t,\ T(x)<1,\ T(y)<1)}{\mathbb{P}(T(x)<1,\ T(y)<1)}$$

$$\overset{ii}{=} \frac{\mathbb{P}(t\leq T(x)<1)\,\mathbb{P}(t\leq T(y)<1)}{{}_1q_x\ {}_1q_y}$$

$$= \frac{(q_x - {}_tq_x)(q_y - {}_tq_y)}{q_x\ q_y}$$

$$= \frac{q_x\ q_y - q_x\,{}_tq_y - {}_tq_x\ q_y + {}_tq_x\,{}_tq_y}{q_x\ q_y}$$

$$= \frac{q_x\ q_y - t\ q_xq_y - t\ q_x\ q_y + t^2 q_x\ q_y}{q_x\ q_y}$$

$$= 1-2t+t^2 \neq 1-t,$$

Hence the PDF is

$$-\frac{d}{dt}(1-2t+t^2) = 2-2t,\ t\in[0,1)$$

.

Cont.

6. You are given that the function $t \mapsto \mu_x(t)$ is constant for $0 \le t < 1$ and that $q_x = 0.16$. Find the value of $t > 0$ such that ${}_tp_x = 0.95$.

Sol

$${}_tp_x = \exp\left\{-\int_0^t \mu_x(s)\,ds\right\} \overset{\text{Cf}\mu}{=} \exp\left\{-\int_0^t \mu_x\,ds\right\} = e^{-\mu_x t},\ t \geq 0$$

$$p_x = 1 - q_x = 0.84 = e^{-\mu_x} \iff \mu_x = -\ln(p_x) = 0.17435$$

So we must solve for t

$$0.95 = e^{-0.17435\,t}$$

$$\iff t = -\frac{1}{0.17435}\ln(0.95) = 0.294197$$

The End.