MID-TERM TEST | MATH 3280 3.00 | OCTOBER 30, 2020

Given name and surname:______________________

Student No:______________________

Signature:______________________

INSTRUCTIONS:

1. Please write everything in **ink**.
2. This exam is a 'closed book' test, duration **60** minutes.
3. Only non-programmable calculators are permitted.
4. There are five questions (each 20%) and a bonus question (10%).

USEFUL FORMULAS:

For $x \geq 0,\ t \in [0,\ 1)$ and $k = 0,\ 1,\ 2,\ \ldots$, if the uniform distribution of deaths assumption holds for the life-status (x), then the following is true

$$_{t+k}p_x \approx (1-t)_k p_x + t_{k+1}p_x.$$

GOOD LUCK!

1. Find to the nearest integer the median of the complete future life-time of a person aged 30 exactly, who is subject to the following force of mortality [redacted]

$$\mu_{30}(t) = \begin{cases} 0.01, & 0 \le t < 10 \\ 0.02, & 10 \le t < 20 \\ 0.03, & 20 \le t \end{cases}$$

Sol.

We must solve $P(T(30) \le m) = 0.5$, $m \in \mathbb{R}_{+}$
where ${}_mp_{30} = \exp\left\{-\int_0^{10} 0.01\,dt - \int_{10}^{20} 0.02\,dt - \int_{20}^{m} 0.03\,dt\right\}$.

Hence $e^{-0.01\cdot 10}\, e^{-0.02\cdot 10}\, e^{-0.03(m-20)} = 0.5$

i.e. $e^{0.3-0.03m} = 0.5$

$\Rightarrow -0.3 + 0.03m = -\ln(0.5) = 0.69315$

$\Rightarrow m = 33.11$

and $\lfloor m \rfloor = 33$

.

Cont.

2. Assume that the future life-times of the life-statuses (x) and (y) are independent and follow Gompertz's law, that is $\mu(t) = a \times b^t,\ t > 0$, where a, b are positive parameters. Find a single life-status (w) that has force of mortality equal to the force of mortality of the joint life-status $(x : y)$ for all $t > 0$.

Sol

We must have $\mu_w(t) = \mu_{x:y}(t)$, $t > 0$.

Hence: $a b^{w+t} = a b^{x+t} + a b^{y+t}$

$\therefore \quad a b^w b^t = a b^t (b^x + b^y)$

$\Rightarrow \quad b^w = b^x + b^y$

$\Rightarrow \quad w = \ln(b^x + b^y) / \ln(b)$

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3. You are given the following excerpt from a life table (the select period is two years). Assume UDD between integer ages and compute ${}_{.90}q_{[60]+0.6}$. [redacted]

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
60	80,625	79,954	78,839	62
61	79,137	78,402	77,252	63
62	77,575	76,770	75,578	64

Sol

We have $\quad {}_{0.9}q_{[60]+0.6} = {}_{0.4}q_{[60]+0.6} + {}_{0.4}p_{[60]+0.6} \cdot {}_{0.5}q_{[60]+1}$

$$= {}_{0.4}q_{[60]+0.6} + \left(1 - {}_{0.4}q_{[60]+0.6}\right) \cdot {}_{0.5}q_{[60]+1}$$

$$= \frac{0.4\, q_{[60]}}{1 - 0.6\, q_{[60]}} + \left(1 - \frac{0.4\, q_{[60]}}{1 - 0.6\, q_{[60]}}\right) 0.5\, q_{[60]+1},$$

where

$$q_{[60]} = 1 - \frac{l_{[60]+1}}{l_{[60]}} = 0.00832248$$

$$q_{[60]+1} = 1 - \frac{l_{62}}{l_{[60]+1}} = 0.01394052$$

Hence:

$${}_{0.9}q_{[60]+0.6} = 0.01029.$$

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4. Let $T'(x)$, $T'(y)$, ξ be independent random variables distributed Weibull with the respective rate parameters $\lambda'_x > 0$, $\lambda'_y > 0$, $\lambda > 0$ and the common shape parameter $\gamma > 0$. Set $T(x) = (T'(x) \wedge \xi)$ and $T(y) = (T'(y) \wedge \xi)$. (Recall that, for instance, $\mathbb{P}(\xi > t) = e^{-(\lambda t)^\gamma}$, $\lambda > 0$, $\gamma > 0$, $t \geq 0$.) Derive the probability $\mathbb{P}(T(x : y) \geq t)$, $t \geq 0$. Use this probability to find $\mu_{x:y}(t) = \mu((x : y) + t)$, $t > 0$.

Sol

$$\mathbb{P}(T(x:y) \geq t) = e^{-(\lambda_x t)^\gamma} e^{-(\lambda_y t)^\gamma} e^{-(\lambda t)^\gamma}$$

$$= \exp\left\{-\left[(\lambda_x t)^\gamma + (\lambda_y t)^\gamma + (\lambda t)^\gamma\right]\right\}$$

$$= \exp\left\{-\left[t^\gamma (\lambda_x^\gamma + \lambda_y^\gamma + \lambda^\gamma)\right]\right\}$$

$$= \exp\left\{-\left[t^\gamma \tilde{\lambda}^\gamma\right]\right\},$$

where $\tilde{\lambda} = (\lambda_x^\gamma + \lambda_y^\gamma + \lambda^\gamma)^{\frac{1}{\gamma}}$. Hence $T(x:y)$ is also distributed Weibull.

Also:

$$-\frac{d}{dt} \exp\{-(t\tilde{\lambda})^\gamma\} = \exp\{-(t\tilde{\lambda})^\gamma\}\tilde{\lambda}\gamma(t\tilde{\lambda})^{\gamma-1}, \quad t > 0$$

and

$$\mu_{x:y}(t) = \tilde{\lambda}\gamma(t\tilde{\lambda})^{\gamma-1}, \quad t > 0$$

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5. A population is subject to the following force of mortality $\mu(t) = e^{0.0002t} - 1,\ t > 0$. Compute the probability that a life now aged 20 exactly dies between ages 60 exact and 70 exact. Also, compute the probability that this same life survives to age 70 exactly. [redacted].

Sol

$${}_tp_x = \exp\left\{-\int_x^{x+t} \mu(s)\,ds\right\} = \exp\left\{-\int_x^{x+t} \left(e^{0.0002s} - 1\right) ds\right\}$$

$$= \exp\left\{-\frac{e^{0.0002(x+t)} - e^{0.0002x}}{0.0002} + t\right\}.$$

So

$${}_{40|10}q_{20} = {}_{40}p_{20}\left(1 - {}_{10}p_{60}\right)$$

$$= 0.0889.$$

and

$${}_{50}p_{20} = 0.6362$$

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6. In a certain population, smokers have a force of mortality twice that of non-smokers For non-smokers, we have ${}_xp_0 = 1 - x/75,\ 0 \le x \le 75$. Compute $\mathring{e}_{55:65}$ for s smoker (55) and a non-smoker (65). [redacted]

Sol

For non-smokers:

$$ {}_tp_x = \frac{75-x-t}{75-x}, \quad 0 \le x < 75-x. $$

Let μ' denote the mortality function for smokers. Then

$$ {}_tp'_x = \exp\left\{-\int_0^t \mu'_x(s)\,ds\right\} = \exp\left\{-2\int_0^t \mu_x(s)\,ds\right\} $$

$$ = \left({}_tp_x\right)^2, \quad t \ge 0. $$

Hence:

$$ \mathring{e}_{65:55} = \int_0^{10} {}_tp_{65}\ {}_tp'_{55} = \int_0^{10} {}_tp_{65} \left({}_tp_{55}\right)^2 dt $$

$$ = \int_0^{10} \left(\frac{10-t}{10}\right)\left(\frac{20-t}{20}\right)^2 dt $$

$$ = 3\,\frac{13}{24}. $$