We have already seen, e.g., the simple joint life status $(x : \overline{n|})$ along with the corresponding future life time r.v.'s $T(x : \overline{n|})$ and $K(x : \overline{n|})$. We shall generalize the notion to the case when $T(\overline{n|}$ is not a degenerate r.v. We shall thus have $T(x : y)$.

Definition 0.1 (Future lifetime of the joint life status)

The r.v. $T(x : y)$ having an (absolutely) continuous c.d.f. is given by

$$T(x : y) := \min(T(x), T(y)) = \begin{cases} T(x), & T(x) \leq T(y) \\ T(y), & T(x) > T(y) \end{cases},$$

with

$${}_tp_{x;y} = \mathbf{P}[T(x) \geq t, T(y) \geq t], \ t \geq 0,$$

and

$${}_tq_{x:y} = {}_tq_x + {}_tq_y - \mathbf{P}[T(x) < t, T(y) < t], \ t \geq 0.$$

Note.

For independent $T(x)$ and $T(y)$, we certainly have that

$$ {}_tp_{x;y} = {}_tp_x \cdot {}_tp_y = 1 - {}_tq_x - {}_tq_y + {}_tq_x \cdot {}_tq_y, \; t \geq 0, $$

and

$$ {}_tq_{x:y} = {}_tq_x + {}_tq_y - {}_tq_x \cdot {}_tq_y, \; t \geq 0. $$

Question.

Can we calculate the p.d.f. of $T(x:y)$? Remember that

$$ {}_tp_{x;y}\mu(x:y+t) = \frac{d}{dt}{}_tq_{x;y} = -\frac{d}{dt}{}_tp_{x:y} \overset{ind}{=} -\frac{d}{dt}{}_tp_x \cdot {}_tp_x. $$

Proposition 0.1

Let $T(x)$ and $T(y)$ be independent future lifetimes of (x) and (y). Then

$$ {}_tp_{x;y}\mu(x:y+t) = {}_tp_x \cdot {}_tp_y \left(\mu(x+t) + \mu(y+t)\right), \; t > 0. $$

Proof.

We have that

$$\begin{aligned}\frac{d}{dt}({}_tp_x \cdot {}_tp_y) &= {}_tp'_x {}_tp_y + {}_tp_x \cdot {}_tp'_y \\ &= -{}_tp_x \mu(x+t) {}_tp_y - {}_tp_y \mu(y+t) {}_tp_x \\ &= -{}_tp_x \cdot {}_tp_y (\mu(x+t) + \mu(y+t)),\end{aligned}$$

which completes the proof. □

Note.

It readily follows that

$$\mu((x:y)+t) \overset{ind}{=} \mu(x+t) + \mu(y+t), \; t > 0.$$

Proposition 0.2

Let $T(x)$ and $T(y)$ be the future lifetimes of (x) and (y). Then

$$ {}_t p_{x:y}\, \mu((x:y)+t) = \int_t^\infty f_{T(x),T(y)}(t,v)dv + \int_t^\infty f_{T(x),T(y)}(u,t)du. $$

Leibniz integral rule

Under certain continuity assumptions, we have that

$$ \frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} \psi(t,v)dv = \frac{d}{dt}\varphi(\alpha(t),\beta(t),t) $$

$$ = \int_{\alpha(t)}^{\beta(t)} \frac{d}{dt}\psi(t,v)dv + \frac{d\beta(t)}{dt}\psi(t,\beta(t)) - \frac{d\alpha(t)}{dt}\psi(t,\alpha(t)) $$

Proof of Proposition 0.2.

We are interested in the derivative

$$f_{T(x:y)}(t) = -\frac{d}{dt}\overline{F}_{T(x:y)}(t) = -\frac{d}{dt}\mathbf{P}[T(x) \geq t, T(y) \geq t],\ t \geq 0.$$

Let

$$\psi(t, v) = \int_t^\infty f_{T(x),T(y)}(u, v)du,$$

we are then interested in the derivative of

$$\overline{F}_{T(x),T(y)}(t,\ t) = \int_t^\infty \int_t^\infty f_{T(x),T(y)}(u, v)dudv = \int_t^\infty \psi(t, v)dv.$$

So we can use Leibniz rule with $\beta(t) = \infty$ and $\alpha(t) = t$, and hence,

$$\frac{d\beta(t)}{dt}\psi(t, \beta(t)) = 0,\ \frac{d\alpha(t)}{dt}\psi(t, \alpha(t)) = \int_t^\infty f_{T(x),T(y)}(u, t)du.$$

Also, using the Leibniz rule again or just due to the FTC,

Proof.

$$\frac{d}{dt}\psi(t,v) = \frac{d}{dt}\int_t^\infty f_{T(x),T(y)}(u,v)du = -f_{T(x),T(y)}(t,v)$$

Hence

$$\begin{aligned} & \frac{d}{dt}\overline{F}_{T(x),T(y)}(t,\ t) \\ = & \ -\left(\int_t^\infty f_{T(x),T(y)}(t,v)dv + \int_t^\infty f_{T(x),T(y)}(u,t)du\right), \end{aligned}$$

that completes the proof. □

Note.

In the case of independent future lifetimes, we have that

$$\int_t^\infty f_{T(x),T(y)}(t,v)dv = f_{T(x)}(t)\int_t^\infty f_{T(y)}(v)dv = f_{T(x)}(t)\overline{F}_{T(y)}(t)$$

and also that

$$\int_t^\infty f_{T(x),T(y)}(u,t)du = f_{T(y)}(t)\int_t^\infty f_{T(x)}(u)du = f_{T(y)}(t)\overline{F}_{T(x)}(t),$$

which gives

$${}_tp_x\mu(x+t){}_tp_y + {}_tp_y\mu(y+t){}_tp_x = {}_tp_x{}_tp_y\left(\mu(x+t)+\mu(y+t)\right),$$

i.e., the p.d.f. of $T(x:y)$ under independence.

Question.

What happens with $K(x)$ and $K(y)$?

Proposition 0.3

The p.n.f. of $K(x:y)$ is

$$\mathbf{P}[K(x:y)=k] = {}_kp_{x:y}\cdot q_{x+k:y+k} = {}_{k|}q_{x:y},\ k=0,1,\ldots.$$

Proof.

We have already proven that for any life status (u), the p.m.f. of $K(u)$ is

$$\mathbf{P}[K(u) = k] = {}_kp_u \cdot q_{u+k}, \; k = 0, 1, \ldots .$$

Thus, for $(u) = (x:y)$, we have that

$$\mathbf{P}[K(x:y) = k] = {}_kp_{x:y} \cdot q_{(x:y)+k}, \; k = 0, 1, \ldots .$$

Moreover, in the case of the joint life status

$$q_{(x:y)+k} = q_{x+k:y+k},$$

which completes the proof. □

Note.

Under independence,

$$\mathbf{P}[K(x:y) = k] = {}_kp_x \cdot {}_kp_y (1 - p_{x+k}p_{y+k}).$$

In other words,

$$\mathbf{P}[K(x:y)=k] = {}_kp_x \cdot {}_kp_y (q_{x+k} + q_{y+k} - q_{x+k}q_{y+k}).$$

Proposition 0.4

The probability that the joint life status will not survive k time units is

$$\mathbf{P}[K(x:y)\leq k] = {}_{k+1}q_{x:y},\ k = 0, 1, \ldots.$$

Note.

The joint life status can be generalized to take into account more than two lives. Then we shall have $T(x_1 : x_2 : \cdots : x_n)$.

Note.

We have already seen the last survivor life status $(\overline{x:y})$. In the case of two lives we conveniently have that

$$T(x:y) + T(\overline{x:y}) = T(x) + T(y).$$

Definition 0.2 (Future lifetime of the last survivor life status)

The r.v. $T(\overline{x:y})$ having an (absolutely) continuous c.d.f. is given by

$$T(\overline{x:y}) := \max(T(x),\ T(y)) = \begin{cases} T(x), & T(x) > T(y) \\ T(y), & T(x) \leq T(y) \end{cases},$$

with

$${}_tq_{\overline{x:y}} = \mathbf{P}[T(x) < t, T(y) < t],\ t \geq 0,$$

and

$${}_tp_{\overline{x:y}} = {}_tp_x + {}_tp_y - \mathbf{P}[T(x) \geq t, T(y) \geq t],\ t \geq 0.$$

These can be further specialized under independence of $T(x)$ and $T(y)$.

Proposition 0.5

Let $(x : y)$ and $\overline{x : y}$ be the joint and last survivor life statuses, respectively. Then

$$ {}_tq_{x:y} + {}_tq_{\overline{x:y}} = {}_tq_x + {}_tq_y, \; t \geq 0. $$

Proof.

Let $\mathcal{A} := \{T(x) < t\}$ and $\mathcal{B} := \{T(y) < t\}$, $t \geq 0$. Then

$$ \mathcal{A} \cap \mathcal{B} = \{T(\overline{x : y}) < t\} $$

and

$$ \mathcal{A} \cup \mathcal{B} = \{T(x : y) < t\}. $$

Then employ

$$ \mathbf{P}[\{\mathcal{A} \cup \mathcal{B}\}] = \mathbf{P}[\{\mathcal{A}\}] + \mathbf{P}[\{\mathcal{B}\}] - \mathbf{P}[\{\mathcal{A} \cap \mathcal{B}\}] $$

to complete the proof. □

Note.

We can use the proposition to obtain, say, ${}_tq_{\overline{x:y}}$. Namely,

$$\begin{aligned} {}_tq_{\overline{x:y}} &= {}_tq_x + {}_tq_y - {}_tq_{x:y} \\ &= {}_tq_x + {}_tq_y - ({}_tq_x + {}_tq_y - \mathbf{P}[T(x) < t, T(y) < t]) \\ &= \mathbf{P}[T(x) < t, T(y) < t],\ t \geq 0. \end{aligned}$$

Note.

As ${}_tq_u = 1 - {}_tp_u$ for any $t \geq 0$ and any life status (u), we have that

$${}_tp_{x:y} + {}_tp_{\overline{x:y}} = {}_tp_x + {}_tp_y,\ t \geq 0.$$

Hence

$${}_tp_{\overline{x:y}} = {}_tp_x + {}_tp_y - \mathbf{P}[T(x) \geq t, T(y) \geq t],\ t \geq 0.$$

Proposition 0.6

The force of mortality of the last survivor life status is

$$\mu(\overline{x:y}+t)=\frac{{}_tp_x\mu(x+t)+{}_tp_y\mu(y+t)-{}_tp_{x:y}\mu((x:y)+t)}{{}_tp_{\overline{x:y}}},$$

where $t>0$.

Proof.

Because of Proposition 0.5, it holds that

$${}_tp_x\mu(x+t)+{}_tp_y\mu(y+t)={}_tp_{x:y}\mu((x:y)+t)+{}_tp_{\overline{x:y}}\mu(\overline{x:y}+t),$$

for $t>0$. Dividing by ${}_tp_{\overline{x:y}}\neq 0$ throughout completes the proof. □

Corollary 0.1

Under independence we have that

$$\mu((\overline{x:y})+t)=\frac{{}_tp_x\cdot{}_tq_y\,\mu(x+t)+{}_tp_y\cdot{}_tq_x\,\mu(y+t)}{{}_tp_x\cdot{}_tq_y+{}_tp_y\cdot{}_tq_x+{}_tp_x\cdot{}_tp_y},$$

where $t>0$.

Proof.

The following completes the proof

$$\begin{aligned}
&\mu(\overline{x:y}+t)\\
=\;&\frac{{}_tp_x\mu(x+t)+{}_tp_y\mu(y+t)-{}_tp_x{}_tp_y(\mu(x+t)+\mu(y+t))}{{}_tp_{\overline{x:y}}}\\
=\;&\frac{{}_tp_x\mu(x+t)+{}_tp_y\mu(y+t)-{}_tp_x{}_tp_y\mu(x+t)-{}_tp_x{}_tp_y\mu(y+t)}{{}_tp_x+{}_tp_y-{}_tp_x\cdot{}_tp_y}\\
=\;&\frac{{}_tp_x(1-{}_tp_y)\mu(x+t)+{}_tp_y(1-{}_tp_x)\mu(y+t)}{{}_tq_x\cdot{}_tp_y+{}_tq_y\cdot{}_tp_x+{}_tp_x\cdot{}_tp_y}.
\end{aligned}$$

Note.

The relation we had for c.d.f.'s and p.d.f.'s of $T(x:y)$ and $T(\overline{x:y})$ does not hold in the case of the force of mortality, i.e.,

$$\mu((x:y)+t)+\mu(\overline{x:y}+t)\neq\mu(x+t)+\mu(y+t).$$

Proposition 0.7

The p.m.f. of $K(\overline{x:y})$ is

$$\mathbf{P}[K(\overline{x:y})=k]={}_kp_x\cdot q_{x+k}+{}_kp_y\cdot q_{y+k}-{}_kp_{x:y}q_{(x+k:y+k)},$$

for $k=0,\ 1,\ \ldots$.

Proof.

Note that Proposition 0.5 does not assume any continuity restrictions. Thus

$${}_kq_x+{}_kq_y={}_kq_{x:y}+{}_kq_{\overline{x:y}},\ k=0,\ 1,\ldots.$$

Proof.

As the above is true for any $k = 0,\ 1, \ldots$, we also have that

$$ {}_{k+1}q_x + {}_{k+1}q_y = {}_{k+1}q_{x:y} + {}_{k+1}q_{\overline{x:y}}. $$

Then, observing that, for any (u),

$$ {}_{k+1}q_u - {}_{k}q_u = {}_{k}p_u \cdot q_{u+k}, $$

we have that

$$ \mathbf{P}[K(\overline{x:y}) = k] = {}_{k}p_x \cdot q_{x+k} + {}_{k}p_y \cdot q_{y+k} - {}_{k}p_{x:y}q_{(x:y)+k}, $$

which along with

$$ q_{(x:y)+k} = q_{x+k:y+k} $$

completes the proof. □

Note.

Under independence and for $k = 0, 1, \ldots,$ we have that

$$\begin{aligned} & \mathbf{P}[K(\overline{x : y}) = k] \\ = \;& {}_kp_x q_{x+k} + {}_kp_y q_{y+k} - {}_kp_x \cdot {}_kp_y (q_{x+k} + q_{y+k} - q_{x+k} q_{y+k}) \\ = \;& {}_kp_x (1 - {}_kp_y) q_{x+k} + {}_kp_y (1 - {}_kp_x) q_{y+k} + {}_kp_x \cdot {}_kp_y q_{x+k} q_{y+k} \\ = \;& {}_kq_y \cdot {}_kp_x q_{x+k} + {}_kq_x \cdot {}_kp_y q_{y+k} + {}_kp_x \cdot {}_kp_y q_{x+k} q_{y+k}. \end{aligned}$$

Question.

Can we find treat the life status $(\overset{1}{x} : y)$ in a similar manner?

Definition 0.3 (Future lifetime of an ordered life status)

The r.v. $T(\overset{1}{x}:y)$ having an (absolutely) continuous c.d.f. is given by

$$T(\overset{1}{x}:y) := \begin{cases} T(x), & T(x) \leq T(y) \\ \infty, & T(x) > T(y) \end{cases},$$

with, for $t \geq 0$,

$${}_tq_{\overset{1}{x}:y} = \mathbf{P}[T(x) < t, T(x) \leq T(y)] = \int_0^t \int_u^\infty f_{T(x),T(y)}(u,s)dsdu.$$

Proposition 0.8

The p.d.f. of the above r.v. is

$$f_{T(\overset{1}{x}:y)}(t) = \int_t^\infty f_{T(x),T(y)}(t,s)ds, \; t \geq 0.$$

Proof.

Let

$$\psi(u) := \int_u^\infty f_{T(x),T(y)}(u,s)ds,$$

and take the derivative

$$\frac{d}{dt}\int_0^t \psi(u)du = \psi(t) = f_{T(\overset{1}{x}:y)}(t).$$

This completes the proof. □

Corollary 0.2

Under independence,

$${}_tq_{\overset{1}{x}:y} = \int_0^t {}_sp_y \cdot {}_sp_x\,\mu(x+s)ds,\ t \geq 0.$$

Proof.

Indeed, for non-negative t, we have that

$$
\begin{aligned}
{}_tq_{\overset{1}{x}:y} &= \int_0^t \int_u^\infty f_{T(x),T(y)}(u,s)dsdu \\
&= \int_0^t \left(\int_u^\infty f_{T(y)|T(x)}(s|u)ds\right) f_{T(x)}(u)du \\
&= \int_0^t \mathbf{P}[(T(y) > u|T(x) = u)]_u p_x \mu(x+u)du \\
&\overset{ind}{=} \int_0^t {}_up_y \cdot {}_up_x \mu(x+u)du,
\end{aligned}
$$

as required. □

Note.

The p.d.f. is easily obtained as:

$$f_{T(\overset{1}{x}:y)}(t) = {}_tp_y \cdot {}_tp_x \mu(x+t), \; t \geq 0.$$

Corollary 0.3

The c.d.f. of the r.v. $T(\overset{1}{x}:\overline{n|})$ is

$$ {}_tq_{\overset{1}{x}:\overline{n|}} = \begin{cases} {}_tq_x, & t \le n \\ {}_nq_x, & t > n \end{cases} \text{ and } {}_tp_{\overset{1}{x}:\overline{n|}} = \begin{cases} {}_tp_x, & t \le n \\ {}_np_x, & t > n \end{cases} . $$

Proof.

Note that, say, for $t > n$,

$$ \begin{aligned} {}_tq_{\overset{1}{x}:\overline{n|}} &= \int_0^n {}_sp_{\overline{n|}} \cdot {}_sp_x \mu(x+s)ds + \int_n^t {}_sp_{\overline{n|}} \cdot {}_sp_x \mu(x+s)ds \\ &= \int_0^n {}_sp_x \mu(x+s)ds = {}_nq_x . \end{aligned} $$

The other part follows in the same fashion, and this completes the proof. □

Question.

Is the probability ${}_tq_{x:\overset{2}{y}}$ the same as ${}_tq_{\overset{1}{x}:y}$?

Definition 0.4 (Future lifetime of an ordered life status)

The r.v. $T(x:\overset{2}{y})$ having an (absolutely) continuous c.d.f. is given by

$$T(x:\overset{2}{y}) := \begin{cases} T(y), & T(x) < T(y) \\ \infty, & T(x) \geq T(y) \end{cases},$$

with

$${}_tq_{x:\overset{2}{y}} = \mathbf{P}[T(x) \leq T(y) \leq t] = \int_0^t \int_u^t f_{T(x),T(y)}(u,s)dsdu, \ t \geq 0.$$

Proposition 0.9

The p.d.f. of $T(x:\overset{2}{y})$ *is*

$$f_{T(x:\overset{2}{y})}(t) = \int_0^t f_{T(x),T(y)}(u,t)du, \ t \geq 0.$$

Proof.

Let

$$\psi(u,t) := \int_u^t f_{T(x),T(y)}(u,s)ds.$$

Then we need to find the derivative

$$\frac{d}{dt}{}_tq_{x:\overset{2}{y}} = \frac{d}{dt}\int_0^t \psi(u,t)du = \psi(t,t) + \int_0^t \frac{\partial}{\partial t}\psi(u,t)du,$$

which results in

$$\frac{d}{dt}{}_tq_{x:\overset{2}{y}} = \int_0^t f_{T(x),T(y)}(u,t)du.$$

Corollary 0.4

Under independence, we have that

$$ {}_nq_{\overset{2}{x:y}} = \int_0^n {}_sp_y \cdot {}_sp_x\mu(x+s)ds -_n p_y \int_0^n {}_sp_x\mu(x+s)ds., $$

for $n \geq 0$. Also,

$$ {}_np_{\overset{2}{x:y}}\mu((x:\overset{2}{y})+n) = {}_np_y\mu(y+n) \cdot {}_nq_x, $$

for $n \geq 0$.

Proof.

For the density, the result follows immediately. Further,

Proof.

Under independence,

$$
\begin{aligned}
{}_nq_{x:\overset{2}{y}} &= \int_0^n \int_s^n f_{T(x),T(y)}(s,t)dtds \\
&= \int_0^n \int_s^n f_{T(y)|T(x)}(t|s) f_{T(x)}(s)dtds \\
&= \int_0^n \mathbf{P}[s < T(y) \leq n | T(x) = s] f_{T(x)}(s)ds \\
&= \int_0^n \mathbf{P}[s < T(y) \leq n | T(x) = s] {}_sp_x \mu(x+s)ds \\
&\overset{ind}{=} \int_0^n \mathbf{P}[s < T(y) \leq n] {}_sp_x \mu(x+s)ds \\
&= \int_0^n ({}_sp_y - {}_n p_y) {}_sp_x \mu(x+s)ds,
\end{aligned}
$$

as required. □

Note.

It readily follows that

$$T(\overset{1}{x}:y) \leq T(x:\overset{2}{y}).$$

Hence

$$\mathbf{P}[T(\overset{1}{x}:y) < t] = {}_t q_{\overset{1}{x}:y} \geq {}_t q_{x:\overset{2}{y}} = \mathbf{P}[T(x:\overset{2}{y}) < t], \; t \geq 0,$$

and also

$${}_t p_{\overset{1}{x}:y} \leq {}_t p_{x:\overset{2}{y}}, \; t \geq 0.$$

Proposition 0.10

The full expectancy of life is

$$\mathring{e}_{x:y} = \int_0^\infty {}_tp_{x:y} dt$$

for the joint life status, and

$$\mathring{e}_{\overline{x:y}} = \int_0^\infty {}_tp_{\overline{x:y}} dt$$

for the last survivor life status. Also, it holds that

$$\mathring{e}_{\overline{x:y}} + \mathring{e}_{x:y} = \mathring{e}_x + \mathring{e}_y.$$

Proof.

Recalling the formula for $\mathring{e}_u$ along with the fact that $T_x + T_y = T_{x:y} + T_{\overline{x:y}}$ completes the proof. □

Proposition 0.11

We have that

$$\mathbf{Var}[T(x:y)] = 2\int_0^\infty t\,{}_t p_{x:y}dt - (\mathring{e}_{x:y})^2,$$

and

$$\mathbf{Var}[T(\overline{x:y})] = 2\int_0^\infty t\,{}_t p_{\overline{x:y}}dt - (\mathring{e}_{\overline{x:y}})^2.$$

Also

$$\mathbf{Cov}[T(x:y), T(\overline{x:y})] = \mathbf{Cov}[T(x), T(y)] + (\mathring{e}_x - \mathring{e}_{x:y})(\mathring{e}_y - \mathring{e}_{x:y}).$$

Proof.

The formulas for the variances are straight forward. As to the covariance:

Proof.

$$
\begin{aligned}
& \mathbf{Cov}[T(x:y), T(\overline{x:y})] \\
= \;& \mathbf{E}[T(x:y)T(\overline{x:y})] - \mathbf{E}[T(x:y)]\mathbf{E}[T(\overline{x:y})] \\
= \;& \mathbf{E}[T(x)T(y)] - \mathbf{E}[T(x:y)]\mathbf{E}[T(\overline{x:y})] \\
= \;& \mathbf{E}[T(x)T(y)] \\
- \;& \mathbf{E}[T(x:y)]\left(\mathbf{E}[T(x)] + \mathbf{E}[T(y)] - \mathbf{E}[T(x:y)]\right).
\end{aligned}
$$

Further, adding and subtracting $\mathbf{E}[T(x)]\mathbf{E}[T(y)]$, we obtain

$$
\begin{aligned}
& \mathbf{Cov}[T(x:y), T(\overline{x:y})] \\
= \;& \mathbf{Cov}[T(x), T(y)] \\
+ \;& \left(\mathbf{E}[T(x)] - \mathbf{E}[T(x:y)]\right)\left(\mathbf{E}[T(y)] - \mathbf{E}[T(x:y)]\right),
\end{aligned}
$$

which completes the proof. □

Corollary 0.5

If $T(x)$ and $T(y)$ are uncorrelated, then

$$\textbf{Cov}[T(x:y), T(\overline{x:y})] = (\mathring{e}_x - \mathring{e}_{x:y})(\mathring{e}_y - \mathring{e}_{x:y}).$$