1

Multi-state transition models for actuarial applications

Actuaries regularly use probability models to analyze situations involving risk. These models often involve some entity and the various states in which it might be—alive or dead, intact or failed, et cetera. This chapter introduces a general type of model that can be applied in many such situations.

Section 1 reviews some models of this type that you've probably already seen and then goes on to describe some practical applications for which those models are inadequate. Section 2 then introduces a more general probability model appropriate for these new cases.

1.1 Introduction

What are multi-state transition models? Probability models that descrbe the random movements of a ***subject*** among various ***states***. Often the subject is a person, but it could just as well be a piece of machinery or a loan contract in whose survival or failure you are interested.

You're probably already familiar with some special cases of such situations.

(1.1) **Example** (basic survival models). In a basic survival model for a status (x)—possibly a person aged x—for which you study the failure time $T(x)$ or $K(x)$, you're considering two states: alive (or, more generally, ***Intact***) and dead (or ***Failed***). Models describe the probability of moving from the State Intact to the State Failed at various points in time.

(1.2) **Example** (multiple-decrement survival models). In multiple-decrement models, you're interested not only in the time of failure of a status (x) as in Example 1.1 but also in which of m causes #1, #2, ..., #m was to blame. Models describe the probabilities of moving from the State Intact to one of the States Failed for Cause #1 or Failed for Cause #2 or ... or Failed for Cause #m at various points in time.

(1.3) **Example** (multiple-life models). In multiple-life models you consider the failure time of complex statuses constructed from other statuses. For example, on a pair of statuses (x) and (y), you might be interested in the joint status $x : y$ that fails when the first of (x) and (y) fails or in the last-survivor status $\overline{x : y}$ that fails when the last of (x) and (y) fails. [More complicated statuses include $\overset{1}{x}{:}y$ that fails at the failure of (x) provided that (x) fails before (y), but these will not be treated here.] Our subject is the pair of statuses, and the possible states are: 1) both are intact, 2) (x) is intact but (y) has failed, 3) (y) is intact but (x) has failed, and 4) both have failed. Models describe the probabilities of moving among these states at various points in time.

All three preceding multi-state transition models share a common characteristic: once the subject leaves a state it cannot return to that state. For instance, in Example 1.1 once the state is Failed it stays Failed forever. But there are important applications in which subjects move back and forth among states, possibly returning to states they have previously left.

(1.4) **Example** (disability). In modeling workers' eligibility for various employee benefits, you might want to consider such states as Active, Temporarily Disabled, Permanently Disabled, and Inactive (which might include retirement, death, and withdrawal—although these could also be used as distinct states). Models describe the probabilities of moving among these various states, including the possibility of moving back and forth between Active and Temporarily Disabled several times.

(1.5) **Example** (driver ratings). In modeling insured automobile drivers' ratings by the insurer, you might want to consider states such as Preferred, Standard, and Substandard. Models describe the probabilities of moving back and forth among these states. [You might also include a state Gone for those no longer insured.]

(1.6) **Example** (Continuing Care Retirement Communities—or CCRC's). In a Continuing Care Retirement Community (CCRC), residents may move among various states such as Independent Living, Temporarily in the Health Center, Permanently in the Health Center, and Gone. Models describe the probabilities of moving among these states at various points in time.

To deal with the sort of applications in these last three examples, actuaries need models that allow for moving back and forth among states. Section 1.2 presents one approach to such models.

1.2 Non-homogeneous Markov Chains

In defining models that allow the subject to move back and forth among various states, I'm going to make some simplifying restrictions and consider only models with:

1) discrete time (meaning that the states are described at times 0, 1, 2, ...);
2) a finite number of states in which the subject may be; and
3) history independence (meaning that the probability distribution of the state of the subject at time $n+1$ may depend on the time n and on the state at time n but does *not* depend on the states at times prior to n).

Such a model is called a ***non-homogeneous Markov Chain***, although the general non-homogeneous Markov Chain does not require restrictions 1) and 2) above. When the probability distribution in 3) does *not* depend on the time n, the model is called a ***homogeneous Markov Chain*** or often simply a Markov Chain. The following definition makes this more precise:

(1.7) **Definition** (non-homogeneous Markov Chain). M is a ***non-homogeneous Markov Chain*** when M is an infinite sequence of random variables $M_0, M_1, \cdots$ with the following properties.

1) M_n denotes the ***state number*** of a subject at time n.
2) Each M_n is a discrete-type random variable over r values (usually $1, 2, \cdots, r$ but sometimes $0, 1, \cdots, m$ with $r = m+1$).
3) The ***transition probabilities***

$$Q_n^{(i,j)} = \Pr[M_{n+1} = j \mid M_n = i \text{ and various other previous values of } M_k]$$
$$= \Pr[M_{n+1} = j \mid M_n = i]$$

are history independent.

If the transition probabilities $Q_n^{(i,j)}$—pronounced "q-sub-n i-to-j"—do not in fact depend on n, then they are denoted by $Q^{(i,j)}$ and the Chain is a ***homogeneous Markov Chain***.

Note that history independence implies the *important* and useful fact that the probability of moving from State $\#i$ to $\#j$ and then to $\#k$ is simply the product of the probability of moving from $\#i$ to $\#j$ with the probability of moving from $\#j$ to $\#k$—that is, successive transitions are independent events.

At this point you should re-examine the Examples in Section 1.1 to see how they can be formulated as non-homogeneous Markov Chains. Here's what you'll find.

(1.8) **Example** (basic survival models as in Example 1.1). Let State #0 be that (x) is Intact and State #1 be that (x) is Failed. [The numbering was chosen so that this single-decrement case is consistent with the multiple-decrement case in the next Example.] You should check that the transition probabilities are $Q_n^{(0,0)} = p_{x+n}$, $Q_n^{(0,1)} = q_{x+n}$, $Q_n^{(1,0)} = 0$, and $Q_n^{(1,1)} = 1$.

(1.9) **Example** (multiple-decrement survival models as in Example 1.2). Let State #0 be that (x) is Intact, and State $\#j$ be that (x) has Failed for Cause $\#j$, for $j = 1, 2, \ldots, m$. You should check that the transition probabilities are $Q_n^{(0,0)} = p_{x+n}^{(\tau)}$, $Q_n^{(0,j)} = q_{x+n}^{(j)}$ for $j = 1, 2, \ldots, m$, $Q_n^{(j,j)} = 1$ for $j = 1, 2, \ldots, m$, and $Q_n^{(i,j)} = 0$ for all other values of i and j.

(1.10) **Example** (multiple-life models as in Example 1.3). Let State #1 be that both (x) and (y) are intact, #2 that (x) is intact but (y) has failed, #3 that (y) is intact but (x) has failed, and #4 that both have failed. Assuming for simplicity that (x) and (y) are independent lives, you should check that the transition probabilities are $Q_n^{(1,1)} = p_{x+n:y+n} = p_{x+n}p_{y+n}$, $Q_n^{(1,2)} = p_{x+n}q_{y+n}$, $Q_n^{(1,3)} = p_{y+n}q_{x+n}$, and $Q_n^{(1,4)} = q_{\overline{x+n:y+n}} = q_{x+n}q_{y+n}$; also $Q_n^{(2,2)} = p_{x+n}$ and $Q_n^{(2,4)} = q_{x+n}$ and similarly for $Q_n^{(3,3)}$ and $Q_n^{(3,4)}$; $Q_n^{(4,4)} = 1$; and all other $Q_n^{(i,j)} = 0$.

(1.11) **Example** (disability as in Example 1.4). Let State #1 stand for the employee's being Active, #2 for Temporarily Disabled, #3 for Permanently Disabled, and #4 for Inactive. Clearly we must have $Q_n^{(3,1)} = Q_n^{(3,2)} = 0$ since #3 denotes *permanent* disability. Unless we wish to model situations allowing a return from the Inactive status, $Q_n^{(4,4)} = 1$ and $Q_n^{(4,j)} = 0$ for $j = 1, 2, 3$. The other transition probabilities would be chosen to reflect observations.

(1.12) **Example** (driver ratings as in Example 1.5). Let State #1 stand for the driver's being classified as Preferred, #2 for Standard, and #3 for Substandard. All the transition probabilities would be chosen to reflect observations, and presumably all could be positive.

(1.13) **Example** (Continuing Care Retirement Communities as in Example 1.6). Let State #1 stand for the resident's being in Independent Living, State #2 for Temporarily in the Health Center, #3 for Permanently in the Health Center, and #4 for Gone. Clearly we must have $Q_n^{(3,1)} = Q_n^{(3,2)} = 0$ since #3 denotes being *Permanently* in the Health Center. Unless we wish to model situations allowing a return from the Gone status, $Q_n^{(4,4)} = 1$ and $Q_n^{(4,j)} = 0$ for $j = 1, 2, 3$. The other transition probabilities would be chosen to reflect observations.

More probabilities

Actuarial notation often uses q to denote failure probabilities [such as moving from State #0 (Intact) to the different State #1 (Failed) in the basic survival model] and p to denote success probabilities [remaining in State #0 in the basic survival model]. Analogously, it is sometimes convenient to use:

(1.14) **Notation.** $P_n^{(i)} = Q_n^{(i,i)}$ is the "success probability" of remaining in State $\#i$ at the next time step.

Even more convenient is to place the probabilities $Q_n^{(i,j)}$ in a matrix:

(1.15) **Definition** (transition probability matrix). The ***transition probability matrix*** $\boldsymbol{Q}_n$ is the r-by-r matrix whose entry in row i and column j—the ***(i,j)-entry***—is the transition probability $Q_n^{(i,j)}$.

Using this notation, the probabilities in Example 1.8, for instance, on the basic survival model could have been written as

$$\boldsymbol{Q}_n = \begin{bmatrix} p_{x+n} & q_{x+n} \\ 0 & 1 \end{bmatrix}$$

The transition probabilities $Q_n^{(i,j)}$ and the transition probability matrix $\boldsymbol{Q}_n$ only provide information about the probability distribution of the state one time step in the future. In practice it is often important to know about longer periods of time—witness the importance of ${}_kp_{x+n}$ versus just p_{x+n} in basic survival models. For non-homogeneous Markov Chains, the corresponding notation is:

(1.16) **Notation.** ${}_kQ_n^{(i,j)} = \Pr[M_{n+k} = j \mid M_n = i]$, with ${}_k\boldsymbol{Q}_n$ used for the r-by-r matrix whose (i, j)-entry is ${}_kQ_n^{(i,j)}$.

For basic survival models, of course, ${}_kp_{x+n}$ can be computed from the one-year probabilities as ${}_kp_{x+n} = p_{x+n}p_{x+n+1}\cdots p_{x+n+k-1}$. The same approach works in our more complicated setting.

(1.17) **Example** (longer-term probabilities). Consider a simple example of a homogeneous Markov Chain with $r = 2$ states #1 and #2 and with transition probability matrix

$$\boldsymbol{Q} = \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}.$$

Suppose that you want to compute ${}_2Q^{(1,2)}$, the probability that the subject, now in State #1, will be in State #2 after two time periods. The subject can be in State #2 after two transitions in either of two ways—by moving $\#1 \to \#1 \to \#2$ or by moving $\#1 \to \#2 \to \#2$. So the probability is the sum of those two probabilities. Thanks to history independence, the events $\#1 \to \#1$ and $\#1 \to \#2$ are independent, and so $\Pr[\#1 \to \#1 \to \#2] = \Pr[\#1 \to \#1]\,\Pr[\#1 \to \#2] = Q^{(1,1)}\,Q^{(1,2)} = 0.4 \times 0.6$. Similarly $\Pr[\#1 \to \#2 \to \#2] = 0.6 \times 0.2$. Thus ${}_2Q^{(1,2)} = 0.4 \times 0.6 + 0.6 \times 0.2$. But—and here is the important observation, so check it—this is the same as the (1,2)-entry of the matrix $\boldsymbol{Q} \times \boldsymbol{Q}$. A similar argument shows that ${}_2Q^{(i,j)}$ is in general the (i,j)-entry of the matrix $\boldsymbol{Q} \times \boldsymbol{Q}$. That is, ${}_2\boldsymbol{Q} = \boldsymbol{Q}^2$.

The argument used in Example 1.17 extends easily to the general case of longer-term probabilities for non-homogeneous Markov Chains, resulting in

(1.18) **Theorem** (longer-term probabilities). In non-homogeneous Markov Chains the longer-term probability ${}_kQ_n^{(i,j)}$ can be computed as the (i,j)-entry of the matrix $\boldsymbol{Q}_n \times \boldsymbol{Q}_{n+1} \times \cdots \times \boldsymbol{Q}_{n+k-1}$—that is,

$$ {}_k\boldsymbol{Q}_n = \boldsymbol{Q}_n \times \boldsymbol{Q}_{n+1} \times \cdots \times \boldsymbol{Q}_{n+k-1}. $$

For a homogeneous Markov Chain, this matrix is just $\boldsymbol{Q}^k$.

Calculation by hand of matrix products can be tedious, even in the Examples below. Fortunately, spreadsheet programs and other mathematical software can perform these calculations easily.

Warning on interpretation: Note that ${}_kQ_n^{(i,j)}$ gives the probability of the subject's being *in* State $\#j$ after k time periods, *not* the probability of *arriving* there exactly k steps in the future. The subject might have reached State $\#j$ previously, left it, returned, *et cetera*. This is of course also true for the special case $i = j$, that is, ${}_kQ_n^{(i,i)}$. The event for which this is the probability allows the subject to have drifted away from State $\#i$, so long as the subject is back again after k time periods—thus, ${}_kQ_n^{(i,i)}$ is not analogous to the smaller "survival" probability of *remaining* in State $\#i$ throughout the k steps. For that, using Notation 1.14 we easily get (check this):

(1.19) **Theorem.** The probability that a subject in State $\#i$ at time n remains in that state through time $n + k$ is

$$ \begin{aligned} {}_kP_n^{(i)} &= \Pr[M_{n+1} = M_{n+2} = \cdots = M_{n+k} = i \,|\, M_n = i] \\ &= P_n^{(i)} P_{n+1}^{(i)} \cdots P_{n+k-1}^{(i)} \end{aligned} $$

(1.20) **Example.** For the homogeneous Markov Chain defined in Example 1.17, let's compute both ${}_2Q_n^{(1,1)}$ and ${}_2P_n^{(1)}$. According to Theorem 1.18 on longer-term probabilities, ${}_2Q_n^{(1,1)}$ is the (1,1)-entry of

$$\boldsymbol{Q}_n\boldsymbol{Q}_{n+1} = \boldsymbol{Q}^2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.36 \\ 0.48 & 0.52 \end{bmatrix},$$

and so ${}_2Q_n^{(1,1)} = 0.64$. From Theorem 1.19 and Notation 1.14, ${}_2P_n^{(1)} = P_n^{(1)}P_{n+1}^{(1)} = Q_n^{(1,1)}Q_{n+1}^{(1,1)} = [Q^{(1,1)}]^2 = (0.4)^2 = 0.16$.

(1.21) **Example.** Consider a Continuing Care Retirement Community (CCRC) with four states: Independent Living, Temporarily in the Health Center, Permanently in the Health Center, and Gone, with the states numbered 1, 2, 3, 4, respectively. Suppose that the transition-probability matrices for a new entrant (at time 0) are as given in the Illustrative Matrices in Section 3.1. Given that this entrant is in Independent Living at time 2, let's find the probability of being there at time 5 and also the probability of remaining there from time 2 through time 5.

The first probability is ${}_3Q_2^{(1,1)}$, which is the (1,1)-entry of $\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4 =$ [note that I only write the entries in the matrices that I actually need in the calculation, writing "–" elsewhere]

$$\begin{bmatrix} 0.60 & 0.15 & 0.15 & 0.10 \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}\begin{bmatrix} 0.50 & 0.20 & 0.20 & 0.10 \\ 0.20 & 0.30 & 0.35 & 0.15 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.40 & - & - & - \\ 0.10 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \end{bmatrix}$$

$$= \begin{bmatrix} 0.60 & 0.15 & 0.15 & 0.10 \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}\begin{bmatrix} 0.22 & - & - & - \\ 0.11 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \end{bmatrix} = \begin{bmatrix} 0.1485 & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}$$

and so the probability ${}_3Q_2^{(1,1)}$ is 0.1485.

The second probability is ${}_3P_2^{(1)} = P_2^{(1)}P_3^{(1)}P_4^{(1)} = Q_2^{(1,1)}Q_3^{(1,1)}Q_3^{(1,1)} = (0.60)(0.50)(0.40) = 0.12$.

(1.22) **Example.** Consider a driver-ratings model in which drivers move among the two classifications Preferred and Standard at the end of each year. Each year: 60% of Preferred are reclassified as Preferred and 40% as Standard; and 70% of Standard are reclassified as Standard and 30% as Preferred. Let's find the probability that a drtiver, known to be classified as Standard at the start of the first year, will be classified as Standard at the start of the fourth year.

Let Preferred be State #1 and Standard be State #2. Then the transition-probability matrix $\boldsymbol{Q}$ for this homogeneous Markov Chain is

$$\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}.$$

We seek ${}_3Q_0^{(2,2)}$, which is the $(2,2)$-entry of $\boldsymbol{Q}^3$. Rather than proceeding as in the preceding Example, consider the following approach. Note that if $\boldsymbol{e}_j$ denotes an $n \times 1$ column matrix with 1 as its j^{th} entry and 0 as its other entries, then for any $k \times n$ matrix $\boldsymbol{M}$ the product $\boldsymbol{M}\boldsymbol{e}_j$ is just the j^{th} column of $\boldsymbol{M}$. Therefore the desired ${}_3Q_0^{(2,2)}$, which is the $(2,2)$-entry of $\boldsymbol{Q}^3$, is just the bottom entry of

$$\boldsymbol{Q}^3\boldsymbol{e}_2 = \boldsymbol{Q}^2(\boldsymbol{Q}\boldsymbol{e}_2) = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.7 \end{bmatrix}$$
$$= \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.52 \\ 0.61 \end{bmatrix} = \begin{bmatrix} 0.556 \\ 0.583 \end{bmatrix},$$

giving 0.583 for the answer.

There's another probability that will prove central to computations in Section 2.2: for a subject in State $\#s$ at time n, the probability of making the transition from State $\#i$ at time $n+k$ to State $\#j$ at time $n+k+1$. In order to possibly make this transition, the subject first must be in State $\#i$ at time $n+k$. Since the subject is now in State $\#s$ at time n, the probability of this is ${}_kQ_n^{(s,i)}$. The probability of the transition then from State $\#i$ to State $\#j$ is $Q_{n+k}^{(i,j)}$. The product ${}_kQ_n^{(s,i)}Q_{n+k}^{(i,j)}$ of these two probabilities gives the probability of the transition in question. That is,

(1.23) **Theorem** (future transition probabilities). Given that a subject is in State $\#s$ at time n, the probability of making the transition from State $\#i$ at time $n+k$ to State $\#j$ at time $n+k+1$ is given by ${}_kQ_n^{(s,i)}Q_{n+k}^{(i,j)}$.

Problems

1. A basic aggregate survival model as in Example 1.1 follows the DeMoivre Law with ultimate age $\omega = 100$. As in Example 1.8, find the matrix $\boldsymbol{Q}_{30}$ for a person aged $x = 60$.
[redacted]

2. Consider a multiple-life model as in Example 1.10 for independent lives aged $x = 60$ and $y = 75$ subject to a DeMoivre Law with $\omega = 100$. As in Example 1.10, find $Q^{(1,2)}$.
[redacted]

3. For the model in Example 1.17, find ${}_3Q^{(2,1)}$.
[redacted]

4. As in Example 1.5, consider a driver-ratings model in which drivers move among the classifications Preferred, Standard, and Substandard at the end of each year. Each year: 60% of Preferreds are reclassified as Preferred, 30% as Standard, and 10% as substandard; 50% of Standards are reclassified as Standard, 30% as Preferred, and 20% as Substandard; and 60% of Substandards are reclassified as Substandard, 40% as Standard, and 0% as Preferred. Find the probability that a driver, classified as Standard at the start of the first year, will be classified as Standard at the start of the fourth year.

[redacted]

5. Consider the situation in Problem 4 again. Find the probability that a driver, classified as Standard at the start of the first year, will be classified as Standard at the start of each of the first four years.

[redacted]

6. Consider the CCRC model in Example 1.21. Find the probability that a resident, in Independent Living at time 1, will not be Gone at time 3.

[redacted]

7. Consider a disability model with four states, numbered in order: Active, Temporarily Disabled, Permanently Disabled, and Inactive. Suppose that the transition-probability matrices for a new employee (at time 0) are as given in the Illustrative Matrices in Section 3.1. For an Active employee at time 1, find the probability the employee is Inactive at time 4.

[redacted]

8. Consider a four-state non-homogeneous Markov Chain with transition probability matrices given by the Illustrative Matrices in Section 3.1. For a subject in State #2 at time 3, find the probability that the subject transitions from State #1 at time 5 to State #3 at time 6.

[redacted]

9. (Theory.) Extend Example 1.17 in general for homogeneous Markov Chains with two states to prove that ${}_2\boldsymbol{Q} = \boldsymbol{Q}^2$.

10. (Theory.) Extend Problem 9 to non-homogeneous Markov Chains with r states to prove that ${}_2\boldsymbol{Q}_n = \boldsymbol{Q}_n\boldsymbol{Q}_{n+1}$.

11. (Theory.) Extend Problem 10 to prove Theorem 1.18 on longer-term probabilities.