Mathematics of Life Contingencies. Math 3280 3.00 F

Instructor: Edward Furman

Homework 7

Unless otherwise indicated, all lives in the following questions are subject to the same law of mortality and their times until death are independent random variables.

1. A multiple decrement model with two causes of decrement has forces of decrement given by

$$\mu_x^{(1)}(t) = \frac{1}{100 - (x+t)}$$

and

$$\mu_x^{(2)}(t) = \frac{2}{100 - (x+t)} \qquad t < 100 - x$$

If x=50, obtain expression for

$a) f_{T,J}(t,j) \qquad b) f_T(t) \qquad c) f_J(j) \qquad d) f_{J|T}(j|t)$

solution

First we obtain ${}_tp_{50}^{(\tau)} = exp[-\int_0^t \frac{3}{50-s}] = (\frac{50-t}{50})^3$

a) $f_{T,J}(t,j) ={}_t p_{50}^{(\tau)} \mu_{50}^{(j)}(t) = (\frac{50-t}{50})^3 (\frac{j}{50-t}) = \frac{j(50-t)^2}{50^3}$

b) $f_T(t) = \sum_{j=1}^2 f_{T,J}(t,j) = \frac{3(50-t)^2}{50^3}$

c) $f_J(j) = \int_0^{50} f_{T,J}(s,j)ds = \frac{j}{50^3} \int_0^{50} (50-s)^2 ds = \frac{j}{50^3}[-\frac{1}{3}(50-s)^3|_0^{50}] = \frac{1}{3}j$

d) $f_{J,T}(j,t) = f_{J,T}(j|t) \cdot f_T(t)$, so $f_{J,T}(j|t) = \frac{f_{J,T}(j,t)}{f_T(t)} = \frac{1}{3}j$.

2. Given the joint p.d.f

$$f_{T,J}(t,j) = pu_1 e^{-(u_1+v_1)t} + (1-p)u_2 e^{-(u_2+v_2)t} \quad 0 \le t, j = 1$$

$$f_{T,J}(t,j) = pv_1 e^{-(u_1+v_1)t} + (1-p)v_2 e^{-(u_2+v_2)t} \quad 0 \le t, j = 2$$

where $0 < p < 1$ and $0 < u_1, u_2, v_1, v_2$.

Find

1) The marginal pdf $f_T(t)$ and $f_J(j)$.

2) The survival function $s_T(t)$.

solution

a)

$$f_T(t) = f_{T,J}(t,1) + f_{T,J}(t,2) = p(u_1+v_1)e^{-(u_1+v_1)t} + (1-p)(u_2+v_2)e^{-(u_2+v_2)t}$$

$$f_T(1) = \int_{t=0}^{\infty} f_{T,J}(t,1)dt = \frac{pu_1}{u_1+v_1} + \frac{(1-p)u_2}{u_2+v_2}$$

$$f_T(2) = \frac{pv_1}{u_1+v_1} + \frac{(1-p)v_2}{u_2+v_2}$$

b)

$$S_T(t) = \int_t^{\infty} f_T(s)ds = pe^{-(u_1+v_1)t} + (1-p)e^{-(u_2+v_2)t}$$

3. For a double-decrement model, you are given the following information:

x	$l_x^{(\tau)}$	$d_x^{(1)}$
30	9480	36
31	9241	72
32	8762	160
33	7699	316
34	5592	445

Calculate the probability that (30) will leave the body of lives within 3 years as a result of decrement (2).

solution We calculate $d_x^{(2)}$

$$d_{30}^{(2)} = 9480 - 9241 - 36 = 203$$

$$d_{31}^{(2)} = 9241 - 8762 - 72 = 407$$

$$d_{32}^{(2)} = 8762 - 7699 - 160 = 903$$

Total is 203+407+903=1513. Dividing by $l_{30}^{(\tau)}$, 1513/9480=0.16.

4. You are given the following triple-decrement table:

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
37	0.05	0.11	0.03
38	0.04	0.03	0.12
39	0.17	0.02	0.09
40	0.08	0.13	0.05
41	0.09	0.04	0.07
42	0.08	0.05	0.06

What is the probability that a life age 37 will remain in the population until age 40 and then leave from cause of decrement number 2 within one year?

solution

$${}_3p_{37}^{(\tau)} = (1-.005-0.11-0.03)(1-0.04-0.03-0.12)(1-0.17-0.02-0.09)$$

$$= (0.81)(0.81)(0.72) = 0.472392$$

$${}_{3|}q_{37}^{(2)} = (0.472392)(0.13) = 0.061411$$

5. In a double decrement table, you are given:

x	$q_x^{(1)}$	$q_x^{(2)}$	$l_x^{(\tau)}$
25	0.01	0.15	
38	0.01	0.10	8400

Calculate the effect on $d_{26}^{(1)}$ if $q_{25}^{(2)}$ change from 0.15 to 0.25.

solution

$l_{25}^{(\tau)} = \frac{8400}{1-0.01-0.15} = 10000$. If 0.15 is changed to 0.25, then $l_{26}^{(\tau)} = 10000(1-0.01-0.25) = 7400$. Currently $d_{26}^{(1)} = 0.01(8400) = 84$; with the higher $q_{25}^{(2)}$, it would be 0.01(7400)=74. The answer is 84-74=10.

6. Insurance policies are subject to decrement by death (decrement 1) or surrender (decrement 2). You are given:

1) $q_{30}^{(1)} = 0.002;\ q_{31}^{(1)} = 0.003;\ q_{32}^{(1)} = 0.004.$

2) $q_{30+k}^{(2)} = c$ for k=0,1,2.

3) ${}_3q_{30}^{(1)} = 0.005.$

Determine c.

solution

Express ${}_3q_{30}^{(1)}$ in terms of c using 1) and 2), and then solve for c.

$${}_3q_{30}^{(1)} = q_{30}^{(1)} + p_{30}^{(\tau)}q_{31}^{(1)} +_2 p_{30}^{(\tau)}q_{32}^{(1)}$$

$$0.005 = 0.002 + (0.998 - c)(0.003) + (0.998 - c)(0.997 - c)(0.004)$$

$$c = 0.428969$$

7. A multiple decrement model has an infinite number of causes of decrement.

   You are given that:

$$p_{40}^{(\tau)} = 0.8$$

$$q_{40}^{(s+1)} = 0.5q_{40}^{(s)} \quad s = 1, 2, \cdots, \infty$$

   Calculate $q_{40}^{(5)}$.

   solution

   The infinite sum $\sum_{t=0}^{\infty} 0.5^t = 2$, and the decrements must sum up to 0.2, so $q_{40}^{(1)} = \frac{0.2}{2} = 0.1$. Then $q_{40}^{(5)} = 0.1(0.5)^4 = 1/160$.

8. Given:

	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$	$q_x^{(\tau)}$
$x < 40$	0.1	0.04	0.02	0.16
$x \geq 40$	0.2	0.04	0.02	0.26

   Calculate ${}_{5|}q_{38}^{(1)}$.

   solution

$$_5p_{38}^{(\tau)} = (1-0.16)^2(1-0.26)^3 = 0.285926$$

$$_{5|}q_{38}^{(1)} =_5 p_{38}^{(\tau)} q_{43}^{(1)}$$

$$= 0.285926(0.2) = 0.057185$$

9. A college student may not finish college due to failure or due to voluntary leaving.

   1) The probability of an entering student staying in college for one year is 0.8.

   2) The probability of an entering student staying in college for two years is 0.7.

   3) The probability that an entering student fails in the first year is twice the probability that a student who completed the first year fails in the second year.

   4) The probability of an entering student voluntarily leaving in the second year is 0.09.

   Calculate the probability of an entering student voluntarily leaving in the first year.

   solution

   Let 1) be failure, 2) be voluntary termination. The probability of an entering student leaving voluntarily in the second year is $_{1|}q_0^{(2)}$, so the probability of a first year student leaving voluntarily is

$$q_1^{(2)} = \frac{_{1|}q_0^{(2)}}{_1p_0^{(\tau)}} = \frac{0.09}{0.8} = 0.1125$$

   We are given that $_2p_0^{(\tau)} = 0.7$, so $p_1^{(\tau)} = \frac{0.7}{0.8} = 0.875$ and $q_1^{(\tau)} = 0.125$. But $q_1^{(\tau)} = q_1^{(1)} + q_2^{(2)}$, so $q_1^{(1)} = 0.125 - 0.1125 = 0.0125$. Then $q_0^{(1)} = 2q_1^{(1)} = 0.025$, and $q_0^{(\tau)} = 1 - 0.8 = 0.2$, so the probability of an entering student voluntarily leaving in the first year, $q_0^{(2)} = 0.2 - 0.025 = 0.175$.

10. Ten years ago, the employees of XYZ inc, all age (60), were given a lifetime health plan.

1) The only sources of decrement are mortality and retirement.

2) Mortality has exactly followed the life table.

3) At age (65) 1,000 of these employees retired and left the pan; there have been no other retirements.

4) No new employees were hired.

5) Today 2,000 employees remain in the program.

How many people were originally in the program?

solution

At age 60, the number of employees was:

1. The group that resulted in the 2000 survivors to age 70, or

$$\frac{2000 l_{60}}{l_{70}} = \frac{2000(8,188,074)}{6,616,155} = 2475$$

2. The group that resulted in the 1000 retirees at age 65, or

$$\frac{1000 l_{60}}{l_{65}} = \frac{1000(8,188,074)}{7,533,964} = 1087$$

So the number of employees at age 60 was 2475+1087=3562.

GOOD LUCK!