MATH 3280 300

Tutorial # 2

Tutorial 2.

Let $T^\circ(x) \sim \mathcal{E}xp(\lambda_x)$, $T^\circ(y) \sim \mathcal{E}xp(\lambda_y)$, $Y \sim \mathcal{E}xp(\lambda)$ be all independent exponentially distributed RVs, and so all parameters are positive real. The distribution of

$$(T(x), T(y)) = (T^\circ(x) \wedge Y, T^\circ(y) \wedge Y)$$

is called "frailty" model.

Ex 1

Find $P[T(x) \geq t]$, $t \geq 0$

Sol

$$P[T(x) \geq t] = P[T^\circ(x) \wedge Y \geq t] = P[T^\circ(x) \geq t, Y \geq t]$$
$$\overset{\perp}{=} P[T^\circ(x) \geq t] P[Y \geq t] = \exp\{-t(\lambda_x + \lambda)\}, \; t \geq 0.$$

Ex 2

Find $P[T(x) \geq t, T(y) \geq s]$, $(t,s) \in (0,\infty)^2$.

Sol

$$P[T(x) \geq t, T(y) \geq s] = P[T^\circ(x) \wedge Y \geq t, T^\circ(y) \wedge Y \geq s]$$
$$= P[T(x) \geq t, T(y) \geq s, Y \geq t \vee s]$$
$$\overset{\perp}{=} P[T(x) \geq t] P[T(y) \geq s] P[Y \geq t \vee s]$$
$$= \exp\{-t\lambda_x - s\lambda_y - (t \vee s)\lambda\}, \; (t,s) \in (0,\infty)^2.$$

Corollary

$$P[T(x) \geq t, T(y) \geq s] \leq P[T^*(x) \geq t, T(y) \geq s]$$

for all $(t, s) \in [0, \infty)^2$.

We say that $(T(x), T(y))$ is Positively Quadrant Dependent (PQD.).

Ex 3

What is $E[T(x)]$?

Sol As $T(x) \sim Exp(\lambda_x + \lambda)$, we have

$$E[T(x)] = \frac{1}{\lambda_x + \lambda} \leq \frac{1}{\lambda_x} = E[T^*(x)]$$

Ex 4

$$P[T(x) = T(y)] > 0.$$

Sol

$$P[T(x) = T(y)] = P[T(x) = Y, T(y) = Y]$$
$$= P[Y \leq T^*(x), Y \leq T^*(y)] > 0$$

Ex 5

What is the PDF of $(T(x), T(y))$?

Sol

Recall that:

$$P[T(x) \geq t, T(y) \geq s] = \exp\{-\lambda_x t - \lambda_y s - (t \vee s)\lambda\}$$

hence, for $t < s$,

$$f(t,s) = \frac{\partial^2}{\partial t \partial s} \exp\{-\lambda_x t - s(\lambda_y + \lambda)\}$$

$$= \exp\{-\lambda_x t - s(\lambda_y + \lambda)\} \lambda_x (\lambda_y + \lambda)$$

and, for $t > s$,

$$f(t,s) = \frac{\partial^2}{\partial t \partial s} \exp\{-t(\lambda_x + \lambda) - \lambda_y s\}$$

$$= \exp\{-\lambda_y s - t(\lambda_x + \lambda)\} \lambda_y (\lambda_x + \lambda)$$

There is $\{T(x) = T(y)\}$ to take care of!

Ex 6

What is $P[T(x) = T(y)]$?

Sol

We know that $\int_0^\infty \int_0^\infty f(t,s)\, dt\, ds + P[T(x) = T(y)] = 1$

$$\int_0^\infty \int_0^\infty f(t,s)\,dt\,ds = \int_0^\infty \int_0^s f(t,s)\,dt\,ds$$

$$+ \int_0^\infty \int_s^\infty f(t,s)\,dt\,ds = ① + ②$$

$$① = \int_0^\infty \int_s^\infty \exp\{-\lambda_y s - t(\lambda_x + \lambda)\}\,\lambda_y(\lambda_x + \lambda)\,dt\,ds$$

$$= \int_0^\infty \left[\int_s^\infty \exp\{-(\lambda_x + \lambda)t\}(\lambda_x + \lambda)\,dt\right] \exp\{-\lambda_y s\}\lambda_y\,ds$$

$$= \int_0^\infty \exp\{-(\lambda_x + \lambda)s\} \exp\{-\lambda_y s\}\lambda_y\,ds$$

$$= \frac{\lambda_y}{\lambda + \lambda_x + \lambda_y}$$

$$② = \frac{\lambda_x}{\lambda + \lambda_x + \lambda_y}$$ similarly.

Hence $P(T(x) = T(y)) = \dfrac{\lambda}{\lambda + \lambda_x + \lambda_y}$.

Ex 6

What is $\mathrm{Cov}[T(x), T(y)]$?

Sol

$$E[T(x)T(y)] = E\left[\int_0^{T(x)}\int_0^{T(y)} dt\, ds\right]$$

$$= \int_0^\infty\int_0^\infty E\left[\mathbb{I}\{T(x) \geq t, T(y) \geq s\}\right] dt\, ds$$

$$= \int_0^\infty\int_0^\infty P(T(x) \geq t, T(y) \geq s)\, dt\, ds$$

$$= \int_0^\infty\int_0^\infty \exp\{-\lambda_x t - \lambda_y s - \lambda(t \vee s)\}\, dt\, ds$$

$$= \int_0^\infty\int_0^s \exp\{-\lambda_x t - \lambda_y s - \lambda s\}\, dt\, ds$$

$$+ \int_0^\infty\int_s^\infty \exp\{-\lambda_x t - \lambda_y s - \lambda t\}\, dt\, ds$$

$$= ① + ②.$$

Here $$② = \int_0^\infty \frac{1}{\lambda_x + \lambda}\exp\{-(\lambda_x + \lambda_y + \lambda)s\}\, ds$$

$$= \frac{1}{(\lambda_x + \lambda)(\lambda_x + \lambda_y + \lambda)}$$

Similarly (1) $= \dfrac{1}{(\lambda_y + \lambda)(\lambda_x + \lambda_y + \lambda)}$

Hence $E(T(x)T(y)) = \dfrac{(\lambda_y + \lambda) + \lambda_x + \lambda}{(\lambda_x + \lambda)(\lambda_y + \lambda)(\lambda_x + \lambda_y + \lambda)}$

$Cov(T(x), T(y))$

$$= \frac{\lambda_x + \lambda_y + 2\lambda}{(\lambda_x + \lambda)(\lambda_y + \lambda)(\lambda_x + \lambda_y + \lambda)} - \frac{1}{(\lambda_x + \lambda)(\lambda_y + \lambda)}$$

$$= \frac{\lambda}{(\lambda_x + \lambda)(\lambda_y + \lambda)(\lambda_x + \lambda_y + \lambda)}.$$

Ex 7

What is $\rho[T(x), T(y)]$?

Sol

$$\rho = \frac{Cov[T(x), T(y)]}{\sqrt{Var[T(x)]}\sqrt{Var[T(y)]}} = \frac{\lambda}{(\lambda + \lambda_x + \lambda_y)}$$

So $\rho[T(x), T(y)] = P[T(x) = T(y)] \in (0, 1)$