Life table

Definition 1.1 (Life table.)

We shall call the distribution of (u) that describes its life time with respect to all existing sources of decrement a multiple decrement life table.

Example 1.1

$[x,\ x+1)$	$q_x^{(1)}$	$q_x^{(2)}$	q_x^{τ}	l_x^{τ}	$d_x^{(1)}$	$d_x^{(2)}$
$[0,\ 1)$	0.02	0.05	0.07	1000	20	50
$[1,\ 2)$	0.03	0.06	0.09	930	27.9	55.8
$[2,\ 3)$	0.04	0.07	0.11	846.3	33.85	59.24
$[3,\ 4)$	0.05	0.08	0.13	753.21	37.66	60.26
$[4,\ 5)$	0.06	0.09	0.15	655.29	39.32	58.98
...	...	...	...	...	...	...

- Looks like we shall need even more notations...

Definition 1.2 (Random number of survivors to age x.)

Let us have a group of l_0^τ new born children. Then, for $\mathbf{1}^\tau\{j\}$ indicating the survival of the new born child number j to age x,

$$L^\tau(x) := \sum_{j=1}^{l_0^\tau} \mathbf{1}^\tau\{j\}$$

denotes the number of children alive at age x. $L^\tau(x)$ is an r.v.

Definition 1.3 (Expected number of survivors to age x.)

For ${}_xp_0^\tau = \mathbf{P}[\mathbf{1}^\tau\{j\} = 1]$ for every $j = 1, \ldots, l_0$, the expectation of $L^\tau(x)$ is

$$l_x^\tau := \mathbf{E}[L^\tau(x)] = \mathbf{E}\left[\sum_{j=1}^{l_0^\tau} \mathbf{1}^\tau\{j\}\right] = l_0^\tau \cdot {}_x p_0^\tau.$$

More generally

The same can be done for any person or life status age y for $y \geq 0$. Namely, the number of people surviving from age y to age x, where $x \geq y$ is

$$l_x^\tau := l_y^\tau \cdot {}_{x-y}p_y^\tau.$$

Definition 1.4

Let ${}_nD^\tau(x) := L^\tau(x) - L^\tau(x+n)$ denote the group of deaths between ages x and $x+n$. We then define the expected number of deaths (out of l_0^τ and between the aforementioned ages)

$${}_nd_x^\tau := \mathbf{E}[{}_nD^\tau(x)] = l_0^\tau({}_xp_0^\tau - {}_{x+n}p_0^\tau) = l_x^\tau - l_{x+n}^\tau.$$

Definition 1.5

We shall define by ${}_nD^i(x)$, the random number of people age 0 that leave the group between the ages x and $x+n$. due to source of decrement i. Then the expected number of such people is

$$\begin{aligned} {}_nd_x^{(i)} &:= \mathbf{E}[{}_nD^i(x)] = l_0^\tau \int_x^{x+n} {}_tp_0^\tau \cdot \mu^{(i)}(t)dt \\ &= l_0^\tau \int_x^{x+n} {}_yp_0^\tau \cdot {}_{t-y}p_y^\tau \cdot \mu^{(i)}(t)dt \\ &= l_y^\tau \int_{x-y}^{x+n-y} {}_sp_y^\tau \cdot \mu^{(i)}(s+y)ds, \end{aligned}$$

for any $y \le x$.

Remark.

We should have that

$$ {}_nD^{\tau}(x) = \sum_{i=1}^{m} {}_nD^{i}(x), $$

for m being the number of sources of decrement and fixed x and n. Thus

$$ {}_nd_x^{\tau} = \sum_{i=1}^{m} {}_nd_x^{(i)}, $$

Definition 1.6

Divide the previous identity by l_x^{τ} throughout, and have that

$$ {}_nq_x^{(i)} := {}_nd_x^{(i)}/l_x^{\tau} \text{ and } \sum_{i=1}^{m} {}_nq_x^{(i)} = {}_nq_x^{\tau} $$

Remember.

For, e.g., two decrements only we have that ${}_tq_x^\tau = {}_tq_x^{(1)} + {}_tq_x^{(2)}$ as we it has been just shown. However, same additive relation does not hold for the p-related functions. Namely,

$$\begin{aligned} {}_tp_x^\tau &= 1 - {}_tq_x^\tau = 1 - {}_tq_x^{(1)} - {}_tq_x^{(2)} \\ &\neq 1 - {}_tq_x^{(1)} + 1 - {}_tq_x^{(2)} \\ &= {}_tp_x^{(1)} + {}_tp_x^{(2)}. \end{aligned}$$

Definition 1.7

The force of decrement due to the i-th source of decrement is defined for a new born child as

$$\mu^i(t) := \frac{1}{{}_tp_0} \frac{d}{dt} {}_tq_0^{(i)}.$$

Proposition 1.1

We have that

$$\mu^\tau(t) = \sum_{i=1}^{m} \mu^i(t).$$

Proof.

Recall that

$${}_tp_0^\tau \cdot \mu^\tau(t) = \frac{d}{dt} q^\tau(t) = \frac{d}{dt} \sum_{i=1}^{m} {}_tq_0^{(i)} = \sum_{i=1}^{m} \frac{d}{dt} {}_tq_0^{(i)} = {}_t p_0^\tau \sum_{i=1}^{m} \mu^i(t),$$

as needed. □

Note that for $\mu^\tau(t)$ to be a legitimate force of mortality, there is no need that

$$\lim_{t\uparrow\infty} \mu^i(t) = \infty \text{ for every } i = 1, \ldots, m.$$

Definition 1.8

Let $(T(u),\ I(u))' := (T,\ I)'$ be a random vector with $T(u)$ standing for the future life time of a status age u, and $I(u)$ denoting the source of decrement. The expression

$$f_{T,\ I}(t,\ i)\Delta t := \mathbf{P}[t < T \leq t + \Delta t, I = i],\ t \in \mathbf{R}_+,\ i = 1, \ldots, m$$

is then interpreted as the probability that a life status will leave in the interval $(u + t,\ u + t + \Delta t]$ due to the i-th source of decrement. Also,

$$f_T(t) := \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathbf{P}[t < T \leq t + \Delta t] = \sum_{i=1}^{m} f_{T,\ I}(t,\ i)$$

and

$$p_I(i) := \mathbf{P}[I = i] = \int_0^\infty f_{T,\ I}(t,\ i) := {}_\infty q_u^{(i)}.$$

Remark.

Note that the density $f_T(t)$ is exactly the one we have encountered in the single decrement life tables, and it is interpreted using Δt as $f_T(t)\Delta t$ is the probability that (u) leaves the life table (all possible decrements are included). Also, $p_I(i)$ is a new object, and it is seen as the probability (u) leaves due to the i-th source of decrement any time. Of course

$$\int_0^\infty f_T(t)dt = 1 \text{ and } \sum_{i=1}^m p_I(i) = 1.$$

Definition 1.9

Similarly to Definition 1.8. we define the probability that (u) leaves due to the i-th source of decrement during the following t years as

$${}_tq_u^{(i)} := \int_0^t f_{T,\,I}(s,\; i)ds,\; i = 1, \ldots, m.$$

Remark.

As when discussing the random survivorship group, we redenote

$$
\begin{aligned}
{}_t q_u^\tau &:= \mathbf{P}[T(u) < t] = \int_o^t f_T(s)ds \\
{}_t p_u^\tau &:= \mathbf{P}[T(u) \geq t] = 1 -_t q_u^\tau \\
\mu^\tau(u+t) &:= \frac{1}{{}_t q_u^\tau} \frac{d}{dt} {}_t q_u^\tau \\
\mu^{(i)}(u+t) &:= \lim_{h \downarrow 0} \frac{1}{h} \mathbf{P}[t < T \leq t+h,\ I = i |\ T \geq t] \\
&= \frac{f_{T,\,I}(t,\ i)}{\mathbf{P}[T \geq t]} = \frac{f_{T,\,I}(t,\ i)}{{}_t p_u^\tau}, \text{ from where} \\
f_{T,\,I}(t,\ i) &:= {}_t p_u^\tau \cdot \mu^{(i)}(u+t)
\end{aligned}
$$

Proposition 1.2

We have that

$$\mu^{\tau}(u+t) = \sum_{i=1}^{m} \mu^{i}(u+t)$$

and

$$ {}_tq_u^{\tau} = \sum_{i=1}^{m} {}_tq_u^{(i)}.$$

Proof.

Indeed

$$\begin{aligned} {}_tq_u^{\tau} &= \int_0^t f_T(s)ds = \int_0^t \sum_{i=1}^{m} f_{T,\,I}(s,\ i)ds \\ &= \sum_{i=1}^{m} \int_0^t f_{T,\,I}(s,\ i)ds = \sum_{i=1}^{m} {}_tq_u^{(i)}, \end{aligned}$$

Proof.

that proves the second expression. Differentiating it throughout completes the proof. □

Proposition 1.3

The conditional density of $I|T$ is

$$f_{I|T}(i|t) = \frac{\mu^{i}(u+t)}{\mu^{\tau}(u+t)}.$$

Proposition 1.4

The p.m.f. of $(K, I)'$ *is* $\mathbf{P}[K = k, I = i] = {}_k p_u^{\tau} \cdot q_{u+k}^{(i)}$, *for fixed* $u \geq 0$, $k \geq 0$, *and* $i = 1, \ldots, m$.

Proof.

$$
\begin{aligned}
\mathbf{P}[K = k, I = i] &= \mathbf{P}[k \leq T < k+1, I = i] \\
&= \mathbf{P}[k < T \leq k+1, I = i] \\
&= \int_k^{k+1} {}_t p_u^{\tau} \cdot \mu^i(u+t)dt \\
&= \int_0^1 {}_{s+k} p_u^{\tau} \cdot \mu^i(u+s+k)ds \\
&= \int_0^1 {}_s p_{u+k}^{\tau} \cdot {}_k p_u^{\tau} \cdot \mu^i(u+s+k)ds \\
&= {}_k p_u^{\tau} \cdot q_{u+k}^{(i)},
\end{aligned}
$$

Proof.

that is true if there is no select periods in the table and thus completes the proof. □

Actuarial mathematics

Multiple life tables

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November 3, 2014

Proposition.

Fix $x \geq 0$, then the r.v.'s $T(x)$ and $I(x)$ are independent if and only if, for $i = 1, \ldots, m(\in \mathbf{N})$ and $t \geq 0$,

$$\mu^{(i)}(x+t) = c_i \mu^{(\tau)}(x+t),$$

with

$$c_i = \int_0^\infty {}_tp_x^\tau \mu^{(i)}(x+t)dt = \mathbf{P}[I = i].$$

Proof.

Obviously $T(x)$ and $I(x)$ are independent iff

$$f_{T,I}(t,i) = f_T(t) f_I(i),$$

where $t \geq 0,\ i = 1, \ldots, m(\in \mathbf{N})$.

Also

$$
\begin{aligned}
f_{T,I}(t,i) &= {}_t p_x^{(\tau)} \mu^{(i)}(x+t) \\
&= {}_t p_x^{(\tau)} c_i \mu^{(\tau)}(x+t) \\
&= {}_t p_x^{(\tau)} \mu^{(\tau)}(x+t) \int_0^\infty {}_t p_x^{\tau} \mu^{(i)}(x+t) dt \\
&= f_T(t) f_I(i),
\end{aligned}
$$

which completes the proof.

Proposition

Let $T^{'(1)}(x)$ and $T^{'(2)}(x)$ be independent, and let also $T(x)$ be independent on $I(x)$, then

$$_tp_x^{'(i)} = \left(_tp_x^{(\tau)} \right)^{c_i}, \; t \geq 0, \; i = 1, \ldots, m (\in \mathbf{N}).$$

Proof

$$\begin{aligned} _tp_x^{'(1)} &= \mathbf{P}[T^{'(1)}(x) \geq t] = \mathbf{P}[T^{'(1)}(x) \geq t, T^{'(2)}(x) \geq 0] \\ &\overset{ind}{=} \mathbf{P}[T^{'(1)}(x) \geq t]\mathbf{P}[T^{'(2)}(x) \geq 0] \\ &\overset{ind}{=} exp\left\{ -\int_0^t c_i \mu^{(\tau)}(x+s)ds \right\} = \left(_tp_x^{(\tau)} \right)^{c_i}, \end{aligned}$$

which completes the proof.

Proposition

Fix $x \geq 0$ and let $t \geq 0$, then the following are equivalent, for $i = 1, \ldots, m (\in \mathbf{N})$:

$$
\begin{aligned}
a.) &\quad {}_tq_x^{(i)} = c_i \cdot {}_tq_x^{(\tau)} \\
b.) &\quad \mu^{(i)}(x+t) = c_i \cdot \mu^{(\tau)}(x+t) \\
c.) &\quad 1 - {}_tq_x^{\prime(i)} = \left(1 - {}_tq_x^{(\tau)}\right)^{c_i}.
\end{aligned}
$$

Poof

$a.) \Rightarrow b.)$ as

$$
\begin{aligned}
& {}_tq_x^{(i)} = c_i \cdot {}_tq_x^{(\tau)} \\
\Rightarrow \quad & \int_0^t {}_sp_x^{(\tau)} \mu^{(i)}(x+s) ds = c_i \int_0^t {}_sp_x^{(\tau)} \mu^{(\tau)}(x+s) ds
\end{aligned}
$$

and after differentiating

$$
\begin{aligned}
& {}_tq_x^{(i)} = c_i \cdot {}_tq_x^{(\tau)} \\
\Rightarrow\ & {}_tp_x^{(\tau)}\mu^{(i)}(x+t) = c_i \cdot {}_tp_x^{(\tau)}\mu^{(\tau)}(x+t) \\
\Rightarrow\ & \mu^{(i)}(x+t) = c_i \cdot \mu^{(\tau)}(x+t), \text{ for } t \geq 0.
\end{aligned}
$$

As for $b.) \Rightarrow c.)$,

$$
\begin{aligned}
& \mu^{(i)}(x+t) = c_i \cdot \mu^{(\tau)}(x+t) \\
\Rightarrow\ & {}_sp_x^{\prime(i)} = \left({}_sp_x^{(\tau)}\right)^{c_i} \\
\Rightarrow\ & 1 - {}_sq_x^{\prime(i)} = \left(1 - {}_sq_x^{(\tau)}\right)^{c_i}, \text{ for } s \geq 0.
\end{aligned}
$$

As for $c.) \Rightarrow b.)$, we have that

$$
\begin{aligned}
& {}_sp_x^{\prime(i)} = \left({}_sp_x^{(\tau)}\right)^{c_i} \\
\Rightarrow \quad & \mu^{(i)}(x+t) = \frac{1}{({}_tp_x^{(\tau)})^{c_i}} c_i \left({}_sp_x^{(\tau)}\right)^{c_i - 1} {}_tp_x^{(\tau)} \mu^{(\tau)}(x+t) \\
\Rightarrow \quad & \mu^{(i)}(x+t) = c_i \mu^{(\tau)}(x+t), \text{ for } t \geq 0.
\end{aligned}
$$

At last, $b.) \Rightarrow a.)$, as

$$
\begin{aligned}
& \mu^{(i)}(x+t) = c_i \mu^{(\tau)}(x+t) \\
\Rightarrow \quad & \int_0^t {}_sp_x^{(\tau)} \mu^{(i)}(x+s) ds = c_i \int_0^t {}_sp_x^{(\tau)} \mu^{(\tau)}(x+s) ds \\
\Rightarrow \quad & {}_tq_x^{(i)} = c_i \cdot {}_tq_x^{(\tau)}, \text{ for } t \geq 0.
\end{aligned}
$$

Corollary

If either the CLT or the UDD assumption holds for each reason of decrement, then

$${}_tq_x^{(i)} = c_i \cdot {}_tq_x^{(\tau)}$$

Proof.

If the CLT holds, then $\mu^{(i)}(x+t) = \mu^{(i)}(x)$ for $x \geq 0$ and $t \in [0,\ 1)$, then

$$\mu^{(i)}(x+t) = \frac{\mu^{(i)}(x)}{\mu^{(\tau)}(x)}\mu^{(\tau)}(x) = c_i \cdot \mu^{(\tau)}(x),$$

and the statement is true by applying the previous proposition. Also, if the UDD holds, then ${}_tq_x^{(i)} = t \cdot q_x^{(i)}$, and therefore

$$ {}_tq_x^{(i)} \overset{UDD}{=} t \cdot q_x^{(i)} = t \cdot \frac{q_x^{(i)}}{q_x^{(\tau)}} \cdot q_x^{(\tau)} \overset{UDD}{=} \frac{q_x^{(i)}}{q_x^{(\tau)}} {}_tq_x^{(\tau)}, $$

leading to

$$ {}_tq_x^{(i)} \overset{UDD}{=} c_i \cdot {}_tq_x^{(\tau)}, $$

and to the required statement by the previous proposition. This completes the proof.

Note

Remember that the separation of say $p_x^{(\tau)}$ into $p_x^{(i)}$ is possible for $p_x^{\prime(i)}$ and $p_x^{(\tau)}$ not zero. If this is true, then an alternative approach is needed. We shall usually assume UDD or CFM in the associated single decrement tables rather than in the multiple decrement ones. The in the UDD case

$$\begin{aligned} q_x^{(i)} &= \int_0^1 {}_tp_x^{(\tau)}\mu^{(i)}(x+t)dt \\ &= \int_0^1 \prod_{j=1,\ j\neq i}^{m} {}_tp_x^{\prime(j)} \cdot {}_tp_x^{\prime(j)}\mu^{(i)}(x+t)dt \\ &\overset{UDD}{=} q_x^{\prime(j)} \int_0^1 \prod_{j=1,\ j\neq i}^{m} (1 - tq_x^{\prime(j)})dt. \end{aligned}$$

De nition 1.1 (Associated single decrement table.)

We shall de ne:

$$ {}_t p_u^{\prime(j)} := \exp\left\{ -\int_0^t \mu^{(j)}(u+s)ds \right\}, $$

and thus

$$ {}_t q_u^{\prime(j)} := 1 - {}_t p_u^{\prime(j)}, $$

for a life status u and $t \geq 0$ and $j = 1, \ldots, m \in \mathbf{N}$.

Note.

Notice that

$$ q^{(j)} \neq q^{\prime(j)}. $$

How do we build a multiple decrement table?

The usual way is to use the associated single decrement functions. Thus, if, say, $p_x'^{(j)}$ are given for $j = 1, \ldots, m$ and all $x = 0, 1, \ldots,$ then we can nd p_x^{τ} and q_x^{τ}. Breaking, e.g., the latter quantity into $q_x^{(j)}$, $j = 1, \ldots, m$ requires additional assumption.

Proposition 1.1 (CFM assumption)

Let the force of the j-th decrement $\mu^{(j)}$, the associated force of decrement $\mu'^{(j)}$ and the total force of decrement μ^{τ} be constant on the interval $[x,\ x+1)$ for all $x = 0,\ 1, \ldots$. Then

$$ {}_sq_x^{(j)} \overset{CFM}{=} \frac{\ln p_x'^{(j)}}{ln\ \prod_{j=1}^{m} p_x'^{(j)}} \cdot \left(1 - \prod_{j=1}^{m} {}_sp_x'^{(j)}\right). $$

Proof.

We have that by assumption

$$\mu^{(j)}(x+t)=\mu^{(j)}(x) \text{ and } \mu^{\tau}(x+t)=\mu^{\tau}(x) \text{ for } t\in[0,\ 1).$$

Thus, for $0\le s<1$, it holds that

$$_sq_x^{(j)} \overset{CFM}{=} \mu^{(j)}(x)\int_0^s {}_tp_x^{\tau}dt=\frac{\mu^{(j)}(x)}{\mu^{\tau}(x)}\int_0^s {}_tp_x^{\tau}\mu^{\tau}(x)dt \overset{CFM}{=} \frac{\mu^{(j)}(x)}{\mu^{\tau}(x)}{}_sq_x^{\tau}.$$

In addition, for $r\in[0,\ 1)$, it holds that

$$-\ln\left({}_rp_x^{\tau}\right) \overset{CFM}{=} r\mu^{\tau}(x) \text{ as well as } -\ln\left({}_rp_x^{\prime(j)}\right) \overset{CFM}{=} r\mu^{(j)}(x)$$

Thus for $r\in(0,\ 1)$, we have that

$$_sq_x^{(j)} \overset{CFM}{=} \frac{\ln\left({}_rp_x^{\prime(j)}\right)}{\ln\left({}_rp_x^{\tau}\right)}{}_sq_x^{\tau},$$

Proof. Cont.

which after taking the limit $r \uparrow 1$ results in

$$_sq_x^{(j)} \overset{CFM}{=} \frac{\ln\left(p_x^{\prime(j)}\right)}{\ln\left(p_x^{\tau}\right)} {}_sq_x^{\tau},$$

that completes the proof. □

Note.

If $p_x^{\prime(j)}$ or p_x^{τ} equal zero in the proposition above, we can not use the formulas.

Proposition 1.2

The UDD assumption Under the UDD assumption, we have that:

$$q_x^{(j)} = \frac{\ln p_x'^{(j)}}{ln \prod_{j=1}^{m} p_x'^{(j)}} \cdot \left(1 - \prod_{j=1}^{m} p_x'^{(j)}\right).$$

Proof.

We in fact assume that, for $t \in [0,\ 1)$, it holds that

$${}_tq_x^{(j)} \overset{UDD}{=} tq_x^{(j)} \text{ and } {}_tq_x^{(\tau)} \overset{UDD}{=} tq_x^{(\tau)}$$

Also,

$${}_tp_x^{\tau}\mu^{(j)}(x+t) = {}_tp_x^{\tau}\frac{1}{{}_tp_x^{\tau}}\frac{\partial}{\partial t}{}_tq_x^{(j)} \overset{UDD}{=} q_x^{(j)}$$

and

□

Proof. Cont.

$$\mu^{(j)}(x+t) \overset{UDD}{=} \frac{q_x^{(j)}}{1 - tq_x^{\tau}}.$$

Thus, we can nd the associated single decrement function as

$$\begin{aligned}
{}_tp_x^{\prime(j)} &= \exp\left\{-\int_0^t \mu^{(j)}(x+s)ds\right. \\
&\overset{UDD}{=} \exp\left\{-\int_0^t \frac{q_x^{(j)}}{1 - sq_x^{\tau}}ds\right\} \\
&= \exp\left\{\frac{q_x^{(j)}}{q_x^{\tau}}\int_0^t d\ln\left(1 - sq_x^{\tau}\right)\right\} \\
&\overset{UDD}{=} \exp\left\{\frac{q_x^{(j)}}{q_x^{\tau}}\ln\left({}_tp_x^{\tau}\right)\right\} = \left({}_tp_x^{\tau}\right)^{q_x^{(j)}/q_x^{\tau}},
\end{aligned}$$

that completes the proof. □

Associated single decrement tables

Creating a multiple decrement table.

We are often given $m \in \mathbf{N}$ simple life tables (i.e., those with single decrement), and we then want to unify them to produce one multiple decrement life table with m sources of decrement.

Definition 1.10

The associated single decrement functions are defined as

$$ {}_tp_u^{\prime(i)} := \exp\left\{-\int_0^t \mu^i(u+s)ds\right\} $$

and

$$ {}_tq_u^{\prime(i)} := 1 - {}_tp_u^{\prime(i)}. $$

Proposition 1.5

We have that

$$ {}_tp_u^{(\tau)} = \prod_{i=1}^{m} {}_tp_u^{\prime(i)}, $$

for $u \geq 0$ and $t \geq 0$.

Proof.

It holds that

$$ \begin{aligned} {}_tp_u^{(\tau)} &= \exp\left\{-\int_0^t \mu^{(\tau)}(u+s)ds\right\} \\ &= \exp\left\{-\int_0^t \sum_{i=1}^{m} \mu^{(i)}(u+s)ds\right\} \\ &= \prod_{i=1}^{m} \exp\left\{-\int_0^t \mu^{(i)}(u+s)ds\right\}, \end{aligned} $$

Proof.

which by definition of the associated functions completes the proof. □

Corollary 1.1

We have that, for $i = 1, \ldots, m$ and non-negative u and t,

$$ {}_tp_u^{\prime(i)} \geq {}_tp_u^{(\tau)}. $$

Also, the above becomes an equality if $m = 1$.

Corollary 1.2

We have that, for $i = 1, \ldots, m$ and non-negative u and t,

$$ {}_tq_u^{\prime(i)} \geq {}_tq_u^{(i)}. $$

Proof.

We have shown that

$$ {}_sp_u^{\prime(i)} \geq {}_s p_u^{(\tau)}. $$

Thus

$$ {}_sp_u^{\prime(i)} \mu^{(i)}(u+s) \geq {}_s p_u^{(\tau)} \mu^{(i)}(u+s). $$

And

$$ \int_0^t {}_sp_u^{\prime(i)} \mu^{(i)}(u+s)ds \geq \int_0^t {}_sp_u^{(\tau)} \mu^{(i)}(u+s)ds = {}_tq_u^{(i)}. $$

The fact that

$$ \int_0^t {}_sp_u^{\prime(i)} \mu^{(i)}(u+s)ds = {}_tq_u^{\prime(i)} $$

follows see below) and thus completes the proof

$$ -\frac{\partial}{\partial t} {}_tp_u^{\prime(i)} = -\frac{\partial}{\partial t} \exp\left\{-\int_0^t \mu^i(u+s)ds\right\} = {}_tp_u^{\prime(i)} \mu^{(i)}(u+t). $$