Mathematics of Life Contingencies. Math 3280 3.00 F

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Homework 5

Unless otherwise indicated, all lives in the following questions are subject to the same law of mortality and their times until death are independent random variables.

1. Under the assuption of uniform distribution of deaths, show that

   a. $\overset{\circ}{e}_x = e_x + \frac{1}{2}$

   b. $\mathbf{Var}(T) = \mathbf{Var}(K) + \frac{1}{12}$

   solution

   a. $\overset{\circ}{e}_x = \mathbf{E}[T] = \mathbf{E}[K+S] = \mathbf{E}[K] + \mathbf{E}[S] = e_x + \frac{1}{2}$

   b. $\mathbf{Var}(T) = \mathbf{Var}(K+S)$

   From the independent of K and S, under the uniform distribution assumption, it follows that

$$\mathbf{Var}(T) = \mathbf{Var}(K) + \mathbf{Var}(S) = \mathbf{Var}(K) + \frac{1}{12}$$

2. Consider a modification of De Moivre's law given by

$$s(x) = (1 - \frac{x}{w})^{\alpha} \qquad 0 \le x < w, \qquad \alpha > 0$$

   Calculate

   a. $\mu(x)$ *b.* $\overset{\circ}{e}_x$

   solution

   This is an important modification of De Moivre's Law, where the force of mortality is multiplied by a constant, and probabilities of survival are taken to the power of the multiplier.

   a. $\mu(x) = -\frac{s'(x)}{s(x)} = \frac{\alpha(1-\frac{x}{w})^{\alpha-1}}{(1-\frac{x}{w})^{\alpha}} = \frac{\alpha}{w-x}$

b. ${}_tp_x = \frac{s(x+t)}{s(x)} = (\frac{w-x-t}{w-x})^\alpha$,

$$\overset{\circ}{e}_x = \int_0^{w-x} {}_tp_x \, dt = \frac{1}{(w-x)^\alpha}(\frac{1}{\alpha+1}(w-x)^{\alpha+1}) = \frac{w-x}{\alpha+1}$$

3. Using life table and an assumption of uniform distribution of deaths in each year of age to find the median of the future lifetime of a person

   a. Age 0 b. Age 50

   solution

   Since the initial size of the cohort is 100,00, in each case we seek an age at which the population size is exactly 50,00.

   a. We see from the table that the desired age lies between 77 and 78. We set up the following equation:

$$l_{77+t} \approx l_{77} - td_{77} = 51599 - 2721t = 50,000$$

   We have t=0.588. Then the median is 77.588.

   b. Since the size of the cohort ate age 50 is 91526, we seek an age at which the population size is 45763. We can see from the table that such age must lie between 79 and 80. We set up an equation:

$$l_{79+t} \approx l_{79} - td_{79} = 46071 - 2891t = 45763$$

   We then have t 0.1065. Thus the median of T is 79.1065-50=29.1065.

4. The recursion formula is to be used to produce tables of compound interest functions. Find u(1), -c(x)/d(x), and 1/d(x) when

   a. $u(x) = \ddot{a}_{\overline{x}|}$

   b. $u(x) = \ddot{s}_{\overline{x}|}$

   solution

   a. Let u(0) $= 0$, $-\frac{c(x)}{d(x)} = 1$, and $\frac{1}{d(x)} = v$.

   b. Let u(0) $= 0$, $-\frac{c(x)}{d(x)} = i+1$, and $\frac{1}{d(x)} = i+1$.

5. Graph $\mu(x+t) \quad 0 < t < 1$, for each of the three assumptions. Also graph the survival function for each assumption.

   On the slides.

6. A population has 30% who are smokers with a constant force of mortality 0.2 and 70% who are non-smokers with a constant force of mortality 0.1.

   Calculate the 75th percentile of the distribution of the future lifetime of an individual selected at random from this population.

   solution

   The population has a mixture distribution; the probability of lifetime greater than x is $0.3e^{-0.2x}+0.7e^{-0.1x}$. We want to set this equal to 0.25, so that there will be a 25% chance of living longer than x and therefore a 75% chance of living less. Let$y = e^{-0.1x}$. Then we have

$$0.3y^2 + 0.7y - 0.25 = 0$$

$$y = 0.31470$$

$$e^{-0.1x} = 0.31470$$

$$x = -11.5614$$

7. For T, the future lifetime random variable for (0):

   a. $w > 70$

   b. ${}_{40}p_0 = 0.6$

   c. $\mathbf{E}[T] = 60$

   d. $\mathbf{E}[min(T,t)] = t - 0.005t^2, \quad 0 < t < 60$

   Calculate the complete expectation of life at 40.

   solution

The complete expectation of life is the complete expectation bounded by t plus the probability of survival to t times the complete expectation of life after t years.

E[T]=E[min(T, 40)] + $\overset{\circ}{e}_{40}({}_{40}p_0)$

$$60 = (40 - 0.005(40^2)) + \overset{\circ}{e}_{40}(0.6)$$

$$\overset{\circ}{e}_{40} = 46.66667$$

8. You are given that $e_{35} = 49$ and $p_{35} = 0.995$.

   If μ_x is doubled for $35 \leq x \leq 36$, what is the revised value of e_{35} ?

   solution

   By using the recursion formula, $e_x = p_x + p_x e_{x+1}$

$$e'_{35} = \frac{49(0.995^2)}{0.995} = 48.755$$

9. For a life age 50, the curtate expectation of life $e_{50} = 20$. For that same life, you are also given that $p_{50} = 0.97$.

   Determine e_{51}.

   solution

   Using the recursion formula, $e_{50} = p_{50} + p_{50}e_{51}$

$$20 = 0.97 + 0.97e_{51}$$

$$e_{51} = 19.6186$$

10. You are given :

    1) $\overset{\circ}{e}_{40} = 75$.

    2) $\overset{\circ}{e}_{60} = 70$.

3) The force of mortality μ_x for x$\in [40, 60]$ is 1/(k-x) for some k.

Determine k.

solution

Let $\mu = \mu_{50}$ be the force of mortality for $x \in [40, 60]$. By the recursive formula,

Let's evaluate ${}_tp_{40}$ for $t \leq 20$

$$ {}_tp_{40} = exp(-\int_0^t \frac{du}{k-(40+u)}) $$
$$ = \frac{k-40-t}{k-40} $$

Then $\overset{\circ}{e}_{40:\overline{20|}}$ is

$$ \overset{\circ}{e}_{40:\overline{20|}} = \int_0^{20} {}_tp_{40} dt $$
$$ = \int_0^{20} \frac{k-40-t}{k-40} dt $$
$$ = \frac{(k-40)^2-(k-60)^2}{2(k-40)} $$

Substituting into the recursive expression,

$$ 75 = \frac{(k-40)^2-(k-60)^2}{2(k-40)} + \frac{70(k-60)}{k-40} $$
$$ k = 146\frac{2}{3} $$

GOOD LUCK!