

MATH 3280

Tutorial

Tutorial 1

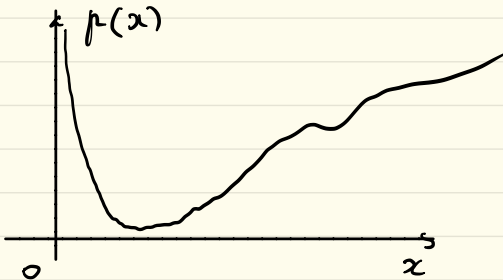
Problem 1

Explore e^{px} as a function of $x \geq 0$.

Solution

$$\begin{aligned}\frac{d}{dx} e^{px} &= \frac{d}{dx} \exp \left\{ - \int_0^x p(x+s) ds \right\} \\&= \exp \left\{ - \int_0^x p(x+s) ds \right\} (-1) \frac{d}{dx} \int_0^x p(x+s) ds \\&= - e^{px} \int_0^x d p(x+s) \\&= - e^{px} [p(x+t) - p(x)] \\&= e^{px} [p(x) - p(x+t)]\end{aligned}$$

Hence e^{px} increases in $x (> 0)$ if $p(x) > p(x+t)$ $t > 0$, and e^{px} decreases in $x (> 0)$ otherwise.



Problem 2

$$\bar{F}(x) = \exp\left\{-\frac{x^3}{12}\right\}, \quad x \geq 0.$$

Is $\bar{F}$ a proper d.d.f.?

Solution

$$\bar{F}(x) > 0 \quad \forall x \geq 0$$

$$\lim_{x \downarrow 0} \bar{F}(x) = 1$$

$$\lim_{x \uparrow \infty} \bar{F}(x) = 0$$

$$\bar{F}'(x) = -\exp\left\{-\frac{x^3}{12}\right\} \frac{x^2}{4} < 0 \quad \forall x \geq 0$$

So, $\bar{F}$ is a proper d.d.f. and so "p".

$$f(x) = -\bar{F}'(x) = \frac{x^2}{4} \exp\left\{-\frac{x^3}{12}\right\}, \quad x > 0$$

$$p(x) = \frac{x^2}{4}, \quad x \geq 0$$

Problem 3

Is $f(t) = x^{n-1} e^{-x/2}$, $x > 0$, $n \geq 1$ a proper p.d.f.

Solution

$$f(t) > 0 \quad \forall t \geq 0$$

$$\int_0^{\infty} f(t) dt = \int_0^{\infty} t^{n-1} e^{-\frac{t}{2}} dt = \Gamma(n) \left(\frac{1}{2}\right)^{-n}$$

$$= \Gamma(n) 2^n$$

So $f(t)$ is not a proper p.d.f

Problem 4

$$p(x) = 0.001 \quad \text{for } 20 \leq x \leq 25, \quad 2/2920 \approx ?$$

Solution

$$u(t)q_x = u p_{x+t} q_{x+t}$$

Hence:

$$2/2920 = 2p_{20} \quad 2q_{22} = 2p_{20} (1 - 2p_{22}) \quad \textcircled{b}$$

Hence

$$2p_{20} = \exp \left\{ - \int_0^2 \mu(20+s) ds \right\} = e^{-2 \cdot 0.001}$$

$$2p_{22} = \exp \left\{ - \int_0^2 \mu(22+s) ds \right\} = e^{-2 \cdot 0.001}$$

$$\textcircled{b} = 0.00199401$$

Problem 5

Let $p(x+t) = t$, $t \geq 0$. Find $\mathbb{E} p(x) p(x+t) \in \mathcal{E}_2$

Solution

$$\begin{aligned}\mathbb{E} p(x) &= \exp \left\{ - \int_0^t p(x+s) ds \right\} \\ &= \exp \left\{ - \int_0^t s ds \right\} = e^{-\frac{t^2}{2}}, \quad t \geq 0.\end{aligned}$$

$$\text{Hence } \mathbb{E} p(x) p(x+t) = t e^{-\frac{t^2}{2}}, \quad t \geq 0$$

Also

$$\begin{aligned}\mathcal{E}_2 &= E[T(x)] = \int_0^\infty e^{-\frac{t^2}{2}} dt = \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{t^2}{2}} dt \\ &= \frac{1}{2} \sqrt{2\pi} = \sqrt{\frac{\pi}{2}}.\end{aligned}$$

Problem 6

Let $\pi(x) \sim \text{Exp}(\lambda)$, $\lambda > 0$. Find $\mathcal{E}_2$, $\text{Var}[T(x)]$, $\text{Var}_{0.5}[T(x)]$

Solution

$$\mathcal{E}_2 = E[T(x)] = \int_0^\infty \lambda e^{-\lambda t} dt = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}$$

$$\text{Var}[T(x)] = \int_0^\infty t^2 \lambda e^{-\lambda t} dt - \mathcal{E}_2^2$$

$$= 2 \frac{1}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\text{VaR}_{0.5}(T(x)) = F_{T(x)}^{-1}(0.5) = -\frac{\ln \frac{1}{2}}{\lambda} = \frac{\ln 2}{\lambda}$$

$$F(t) = 1 - e^{-\lambda t} = q$$

$$\Leftrightarrow \frac{-\ln(1-q)}{\lambda} = t = F^{-1}(q)$$

$$\text{Generally } \text{VaR}_q(X) = \inf \{x \in \mathbb{R} : P[X \leq x] \geq q\}$$

