MATH 328D

Tutorial

Tutorial 1

Problem 1

Explore ${}_t p_x$ as a function of $x \geq 0$.

Solution

$$\frac{\partial}{\partial x}\,{}_t p_x = \frac{\partial}{\partial x} \exp\left\{-\int_0^t \mu(x+s)\,ds\right\}$$

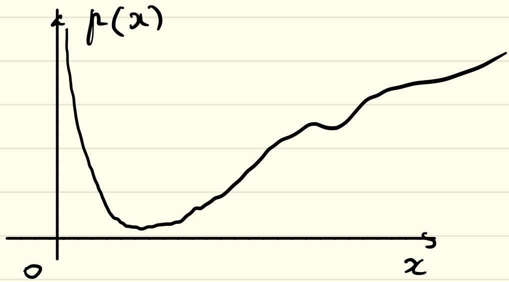
$$= \exp\left\{-\int_0^t \mu(x+s)\,ds\right\}(-1)\frac{\partial}{\partial x}\int_0^t \mu(x+s)\,ds$$

$$= -\,{}_t p_x \int_0^t d\,\mu(x+s)$$

$$= -\,{}_t p_x \left[\mu(x+t) - \mu(x)\right]$$

$$= {}_t p_x \left[\mu(x) - \mu(x+t)\right]$$

Hence ${}_t p_x$ increases in $x\,(>0)$ if $\mu(x) > \mu(x+t)$ $\forall t > 0$, and ${}_t p_x$ decreases in $x\,(>0)$ otherwise.



Problem 2

$$\bar{F}(x) = \exp\{-x^3/12\}, \quad x \geq 0.$$

Is $\bar{F}$ a proper d.d.f.?

Solution

$\bar{F}(x) > 0 \quad \forall x \geq 0$

$\lim_{x \downarrow 0} \bar{F}(x) = 1$

$\lim_{x \uparrow \infty} \bar{F}(x) = 0$

$\bar{F}'(x) = -\exp\{-\frac{x^3}{12}\}\frac{x^2}{4} < 0 \quad \forall x \geq 0$

So, $\bar{F}$ is a proper d.d.f. and so "p".

$f(x) = -\bar{F}'(x) = \frac{x^2}{4}\exp\{-\frac{x^3}{12}\}, \quad x > 0$

$p(x) = \frac{x^2}{4}, \quad x \geq 0$

Problem 3

Is $f(t) = x^{n-1} e^{-x/2}$, $x > 0$, $n \geq 1$ a proper p.d.f.

Solution

$f(t) > 0 \quad \forall t \geq 0$

$$\int_0^{\infty} f(t)\,dt = \int_0^{\infty} t^{n-1} e^{-\frac{t}{2}}\,dt = \Gamma(n)\left(\frac{1}{2}\right)^{-n}$$

$$= \Gamma(n)\, 2^{n}$$

So $f(t)$ is <u>not</u> a proper p.d.f.

Problem 4

$\mu(x) = 0.001$ for $20 \le x \le 25$, ${}_{2|2}q_{20} = ?$

Solution

w/t $q_x = {}_{u}p_x \cdot {}_{t}q_{x+u}$

Hence:

$${}_{2|2}q_{20} = {}_{2}p_{20}\ {}_{2}q_{22} = {}_{2}p_{20}\left(1 - {}_{2}p_{22}\right) = (b)$$

Here

$${}_{2}p_{20} = \exp\left\{-\int_0^2 \mu(20+s)\,ds\right\} = e^{-2\cdot 0.001}$$

$${}_{2}p_{22} = \exp\left\{-\int_0^2 \mu(22+s)\,ds\right\} = e^{-2\cdot 0.001}$$

$(b) = 0.0019901$

Problem 5

Let $\mu(x+t) = t$, $t \geq 0$. Find ${}_tp_x\,\mu(x+t)$ & $\mathring{e}_x$

Solution

$${}_tp_x = \exp\left\{-\int_0^t \mu(x+s)\,ds\right\}$$
$$= \exp\left\{-\int_0^t s\,ds\right\} = e^{-\frac{t^2}{2}}, \quad t \geq 0.$$

Hence ${}_tp_x\,\mu(x+t) = t e^{-\frac{t^2}{2}}$, $t \geq 0$

Also

$$\mathring{e}_x = E[T(x)] = \int_0^\infty e^{-\frac{t^2}{2}}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,dt$$

$$= \frac{1}{2}\sqrt{2\pi} = \sqrt{\frac{\pi}{2}}.$$

Problem 6

Let $T(x) \sim Exp(\lambda)$, $\lambda > 0$. Find $\mathring{e}_x$, $Var[T(x)]$, $VaR_{0.5}[T(x)]$

Solution

$$\mathring{e}_x = E[T(x)] = \int_0^\infty t\lambda e^{-\lambda t}\,dt = \int_0^\infty e^{-\lambda t}\,dt = \frac{1}{\lambda}$$

$$Var[T(x)] = \int_0^\infty t^2\lambda e^{-\lambda t}\,dt - \mathring{e}_x^{\,2}$$
$$= 2\frac{1}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\mathrm{VaR}_{0.5}[T(x)] = F^{-1}_{T(x)}(0.5) = -\frac{\ln \frac{1}{2}}{\lambda} = \frac{\ln 2}{\lambda}$$

$$F(t) = 1 - e^{-\lambda t} = q$$

$$\Leftrightarrow \frac{-\ln(1-q)}{\lambda} = t = F^{-1}(q)$$

Generally $\mathrm{VaR}_q[X] = \inf\{x \in \mathbb{R} : P[X \leq x] \geq q\}$