Life table

Definition 1.1 (Life table.)

We shall call the distribution of (u) its life table.

Example 1.1

$[x,\ x+t)$	l_x	q_x	d_x	L_x	T_x	$\overset{\circ}{e}_x$
$[0,\ 1)$	$100,000$	0.01260	$1,260$	$98,973$	$7,387,758$	73.88
$[1,\ 2)$	$98,740$	0.00093	92	$98,694$	$7,288,785$	73.82
$[2,\ 3)$	$98,648$	0.00065	64	$98,617$	$7,190,091$	72.89
$[3,\ 4)$	$98,584$	0.00050	49	$98,560$	$7,091,474$	71.93
$[4,\ 5)$	$98,535$	0.00040	40	$98,515$	$6,992,914$	70.97
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

- Looks like we shall need more notations...

Definition 1.2 (Random number of survivors to age x.)

Let us have a group of l_0 new born children. Then, for $\mathbf{1}\{j\}$ indicating the survival of the new born child number j to age x,

$$L(x) := \sum_{j=1}^{l_0} \mathbf{1}\{j\}$$

denotes the number of children alive at age x. $L(x)$ is an r.v.

Definition 1.3 (Expected number of survivors to age x.)

For ${}_xp_0 = \mathbf{P}[\mathbf{1}\{j\} = 1]$ for every $j = 1, \ldots, l_0$, the expectation of $L(x)$ is

$$l_x := \mathbf{E}[L(x)] = \mathbf{E}\left[\sum_{j=1}^{l_0} \mathbf{1}\{j\}\right] = l_0 \cdot {}_xp_0.$$

(Think of the binomial r.v.)

Definition 1.4

Let ${}_nD(x) := L(x) - L(x+n)$ denote the group of deaths between ages x and $x+n$. We then define the expected number of deaths (out of l_0 and between the aforementioned ages)

$$ {}_nd_x := \mathbf{E}[{}_nD(x)] = l_0({}_xp_0 - {}_{x+n}p_0) = l_x - l_{x+n}. $$

Proposition 1.1

We have that

$$ \mu(x) = -\frac{1}{l_x} \cdot \frac{d}{dx} l_x. $$

Proof.

Noticing that ${}_xp_0 = l_x/l_0$ completes the proof. □

At home.

Check that

$$l_{x+n} = l_x \cdot \exp\left\{-\int_x^{x+n} \mu(s)ds\right\},$$

$$l_x - l_{x+n} = \int_x^{x+n} l_s \mu(s)ds.$$

Proposition 1.2

We have that the local extrema points of $l_x \mu(x)$ correspond to the points of inflection of l_x.

Proof.

$$\frac{d}{dx} l_x \mu(x) = -\frac{d}{dx} l_x \frac{1}{l_x} \frac{d}{dx} l_x = -\frac{d^2}{dx^2} l_x,$$

which completes the proof. □

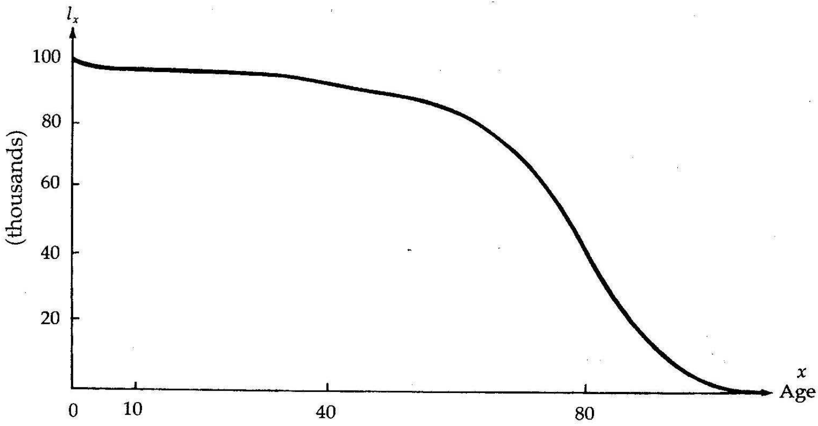

Figure: Plot of l_x

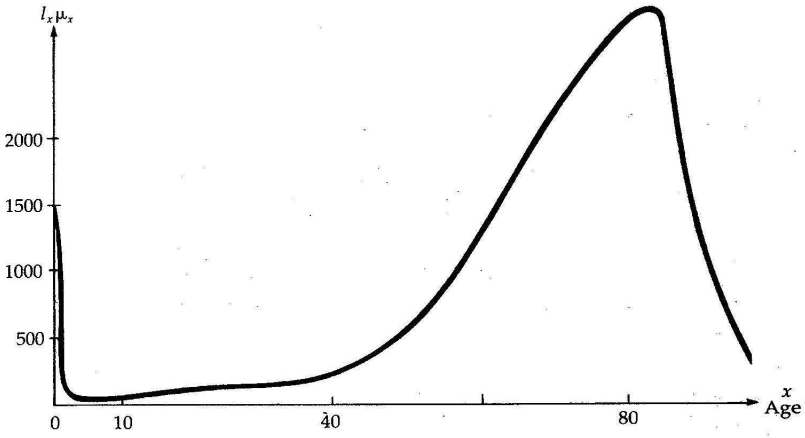

Figure: Plot of $l_x \mu(x)$

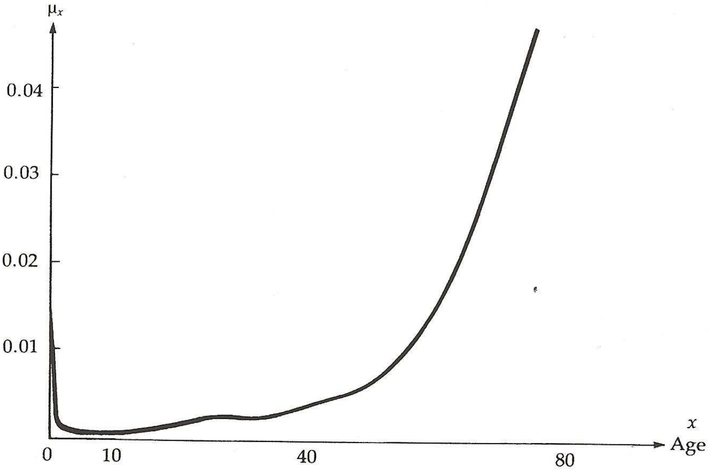

Figure: Plot of $\mu(x)$

Approximating life functions at fractional ages

- Life table functions investigated hitherto specify the c.d.f. of $K(x)$ completely. To specify the c.d.f. of $T(x)$ we must postulate an analytic form or adopt an assumption in addition to the life table functions we have had.
- We shall further review three different assumptions for fractional ages, given a fixed $x = 0, 1, \ldots$ and $t \in (0,\ 1)$,
  1. Linear interpolation or the uniform distribution of deaths (UDD),
  $$S(x+t) = (1-t)S(x) + tS(x+1).$$
  2. Exponential interpolation or the constant force of mortality (CFM),
  $$\log S(x+t) = (1-t)\log S(x) + t\log S(x+1).$$
  3. Harmonic interpolation,
  $$1/S(x+t) = (1-t)/S(x) + t/S(x+1).$$

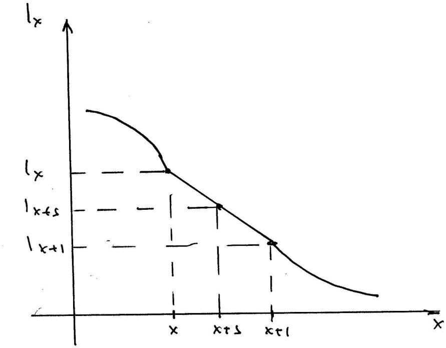

Figure: Linear interpolation for $l_{x+s}, 0 < s < 1$

The UDD

Linear approximation applied.

We find the value of l_{x+s}, $x = 0, 1, 2 \dots$ and $s \in (0,\ 1)$ from the following equations:

$$\frac{l_x - l_{x+s}}{l_x - l_{x+1}} = \frac{x + s - x}{x + 1 - x} = s,$$

which yields

$$l_{x+s} = l_x - sl_x + sl_{x+1}.$$

Finally, we find that:

$$l_{x+s} = (1 - s)l_x + sl_{x+1}.$$

In terms of the number of deaths, we have that

$$l_{x+s} = (1-s)l_x + sl_{x+1} = l_x - sd_x,$$

where $d_x = l_x - l_{x+1}$.

The d.d.f.

Further, dividing by l_x, we have that

$$_sp_x \overset{UDD}{=} 1 - sq_x \Leftrightarrow {}_sq_x \overset{UDD}{=} sq_x.$$

As q_x is tabulated we can calculate $_sq_x$ for any non-integer duration s.

The p.d.f.

Also, we have that

$$f_{T(x)}(s) = \frac{d}{ds}{}_sq_x \overset{UDD}{=} q_x, \text{ for } 0 < s < 1.$$

- As $f_{T(x)}(s)$ is constant in s and equal to q_x, deaths are said to be uniformly distributed over the interval $[x, x+1)$.

We have seen that

$$\mu(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{{}_xp_0} \text{ and, similarly, } \mu(x+s) = \frac{f_{T(x)}(s)}{{}_sp_x}.$$

The force of mortality.

Then the force of mortality is

$$\mu(x+s) \overset{UDD}{=} \frac{q_x}{1 - s \cdot q_x},$$

which increases in s.

Age is fractional as well.

If both the age and the duration are non-integer, i.e., we want to calculate ${}_{s-t}q_{x+t}$, $0 < t < s < 1$, then

$$ {}_sp_x = {}_tp_x \cdot {}_{s-t}p_{x+t} \Leftrightarrow {}_{s-t}p_{x+t} = \frac{{}_sp_x}{{}_tp_x}. $$

Hence,

$$ {}_{s-t}q_{x+t} = 1 - {}_{s-t}p_{x+t} = 1 - \frac{{}_sp_x}{{}_tp_x} = 1 - \frac{1 - {}_sq_x}{1 - {}_tq_x}, $$

which after applying the UDD assumption reduces to

$$ {}_{s-t}q_{x+t} \overset{UDD}{=} 1 - \frac{1 - s \cdot q_x}{1 - t \cdot q_x} = \frac{(s-t)q_x}{1 - t \cdot q_x}, \text{ for } 0 < t < s < 1. $$

Graphically, the main ideas of UDD can be seen as

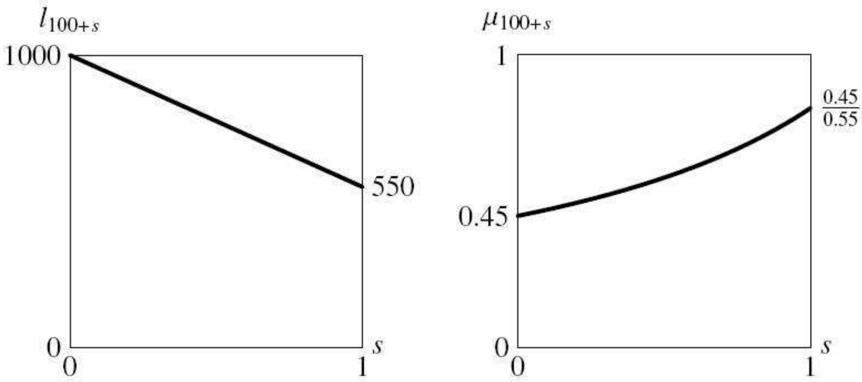

Figure: l_{x+s} decreases linearly and μ_{x+s} increases.

Constant force of mortality

- The UDD approximation assumes a certain (linear) form of ${}_tp_x,\ t \in [0,\ 1)$. Another way to assume specific behavior of ${}_tp_x$ is to use its connection with the force of mortality function. Namely:

CFM

For a fixed non-negative integer x and $t \in [0,\ 1)$, let $\mu(x+t) \equiv \mu(x)$ for all $t \in [0,\ 1)$. Then

$${}_tp_x = \exp\left(-\int_0^t \mu(x+s)ds\right) \overset{CFM}{=} \exp(-t \cdot \mu(x)) = (p_x)^t,$$

and, we also similarly have that

$$p_x = \exp\left(-\int_0^1 \mu(x+s)ds\right) \overset{CFM}{=} e^{-\mu(x)} \Leftrightarrow \mu(x) = -\ln(p_x).$$

Both age and duration are non-integer

Fix $0 \le t < s < 1$, then, for $u = t + r$,

$$
\begin{aligned}
{}_{s-t}p_{x+t} &= \exp\left(-\int_0^{s-t} \mu(x+t+r)dr\right) \\
&= \exp\left(-\int_t^{s} \mu(x+u)du\right) \\
&\overset{CFM}{=} \exp\left(-\int_t^{s} \mu(x)du\right) \\
&= \exp(-(s-t)\mu(x)) = (p_x)^{s-t}.
\end{aligned}
$$

At home.

Due to the above, we have that only the duration is important, i.e.,

$$
{}_{s-t}p_{x+t} = {}_{s-t}p_x.
$$

Why?

At home.

Show that the CFM is consistent with the formula for the exponential interpolation.

Graphically, the main ideas of CFM can be seen as

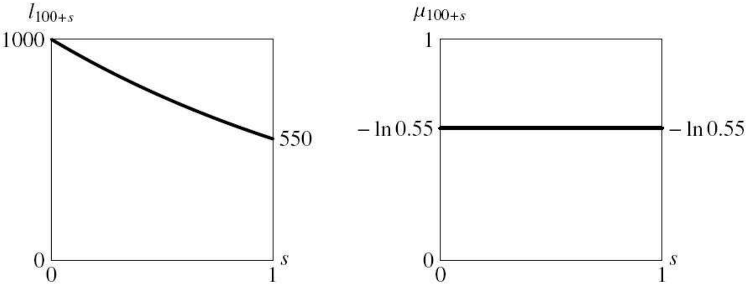

Figure: l_{x+s} decreases and μ_{x+s} is constant.

Proposition 1.1

Let $J(u)(:= J)$ denote an r.v. representing the number of fractional time units (u) lived during the year of death. Then under the CFM the r.v.'s K and $J)$ are generally dependent, while they are independent under the UDD.

Proof.

By definition,

$$\begin{aligned}
\mathbf{P}[K = k,\ J < j] &= \mathbf{P}[k \leq T < k + j] = \mathbf{P}[k < T \leq k + j] \\
&= {}_k p_u \cdot {}_j q_{u+k} \overset{UDD}{=} {}_k p_u \cdot j \cdot q_{u+k} \\
&= {}_{k|} q_u \cdot j = \mathbf{P}[K(u) = k] \cdot \mathbf{P}[J < k],
\end{aligned}$$

which completes the proof for the UDD case. For the CFM case, ${}_j q_{u+k} = 1 - {}_j p_{u+k} = 1 - (p_{u+k})^j$, which together with the above, do not separate into a product. □

Remark.

Note that if $p_{u+k} = p_u$, then ${}_kp_u = (p_u)^k$, and then

$$\begin{aligned}
\mathbf{P}[K = k,\ J < j] &= \mathbf{P}[k \le T < k + j] = \mathbf{P}[k < T \le k + j] \\
&= {}_kp_u \cdot {}_jq_{u+k} \overset{CFM}{=} {}_kp_u \cdot (1 - (p_{u+k})^j) \\
&= (p_u)^k \cdot (1 - (p_{u+k})^j) \\
&= \frac{(p_u)^k(1 - p_u)(1 - (p_{u+k})^j)}{1 - p_u} \\
&= \mathbf{P}[K = k] \cdot \mathbf{P}[J < j].
\end{aligned}$$

Remember.

The UDD assumption implies that the r.v. J is uniform on $[0,\ 1]$. On the other hand, the CFM assumes that J is conditionally exponential with μ, i.e., if $X|X \le 1 \backsim Exp(\mu)$, then J and X have same distribution. Are the r.v.'s equal?

The Balducci assumption

Balducci approximation.

We have that, for $0 \geq t < 1$,

$$\frac{1}{l_{x+t}} \stackrel{Hyp}{=} \frac{1-t}{l_x} + \frac{t}{l_{x+1}} = \frac{(1-t)l_{x+1} + tl_x}{l_x \cdot l_{x+1}},$$

which implies that

$$l_{x+t} \stackrel{Hyp}{=} \frac{l_x \cdot l_{x+1}}{l_{x+1} + td_x} = \frac{l_{x+1}}{p_x + tq_x},$$

and hence, dividing by l_x throughout,

$${}_tp_x \stackrel{Hyp}{=} \frac{p_x}{p_x + tq_x} = \frac{p_x}{1-(1-t)q_x},$$

and

$$ {}_tq_x = \frac{1-(1-t)q_x - p_x}{1-(1-t)q_x} = \frac{tq_x}{1-(1-t)q_x}. $$

Moreover, for $0 \le t < s < 1$, we have that

$$ {}_{s-t}p_{x+t} = \frac{{}_sp_x}{{}_tp_x} \overset{Hyp}{=} \frac{1-(1-t)q_x}{1-(1-s)q_x}. $$

as well as that

$$ {}_{s-t}q_{x+t} = 1 - {}_{s-t}p_{x+t} \overset{Hyp}{=} \frac{q_x(s-t)}{1-(1-s)q_x}. $$

The force of mortality can be found from

$$ \mu_{x+t} = -\frac{d}{dt} \ln {}_tp_x \overset{Hyp}{=} -\frac{d}{dt} \ln \left(\frac{p_x}{1-(1-t)\,q_x} \right), $$

that is

$$\mu_{x+t} \overset{Hyp}{=} \frac{d}{dt} \ln(1-(1-t)\,q_x) = \frac{q_x}{1-(1-t)\,q_x},$$

which decreases in t.

Graphically, the main ideas of Balducci's assumption can be seen as

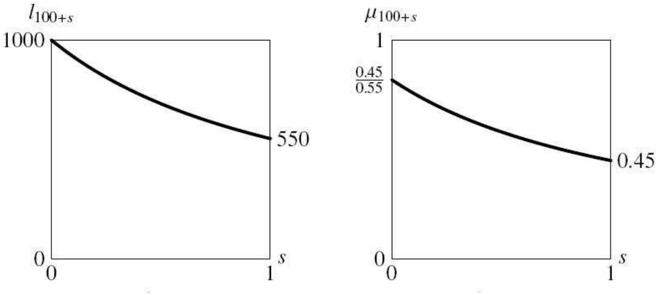

Figure: l_{x+s} decreases and μ_{x+s} decreases.

To conclude the fractional ages

Remember.

For integer non-negative x and t and s in $[0, 1)$, we have that

	UDD	CFM	Balducci
l_{x+s}	$l_x - sd_x$	$l_x \cdot (p_x)^s$	$\frac{l_{x+1}}{p_x+sq_x}$
${}_sq_x$	sq_x	$1-(p_x)^s$	$\frac{sq_x}{1-(1-s)q_x}$
${}_sq_{x+t}$	$\frac{sq_x}{1-tq_x}$	$1-(p_x)^s$	$\frac{sq_x}{1-(1-(s+t))q_x}$
$\mu(x+s)$	$\frac{q_x}{1-sq_x}$	$-\log p_x$	$\frac{q_x}{1-(1-s)q_x}$

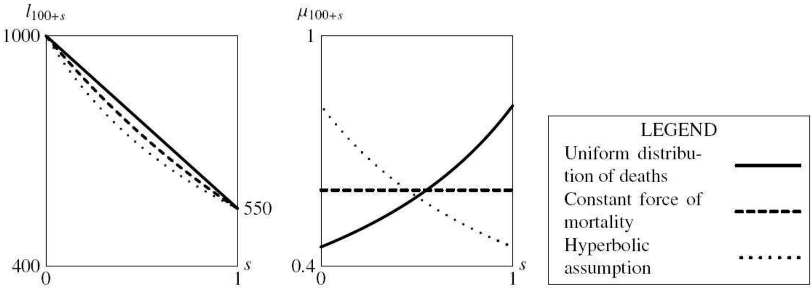

Figure: Fractional ages' concluding plot.

- De Moivre (1725) used a one parameter formula:

$$\mu(x) = (\omega - x)^{-1}, \quad S(x) = 1 - x/\omega, \quad 0 \leq x < \omega.$$

- Gompertz (1825) used a two parameter formula:

$$\mu(x) = Bc^x, S(x) = exp\left(-\frac{B}{\log(c)}(c^x - 1)\right), B > 0, c > 1, x \geq 0.$$

- Makeham (1867) used the following:

$$\mu(x) = A + bc^x, S(x) = exp\left(-Ax - \frac{B}{\log(c)}(c^x - 1)\right),$$

where $B > 0, A \geq -B, c > 1, x \geq 0$.

- Weibull (1951) suggested:

$$\mu(x) = Ax^B, S(x) = exp\left(-A\frac{x^{B+1}}{B+1}\right), A > 0, B > 0, x \geq 0.$$

Proposition 1.2

Let $T(u) \backsim F_u$. Then

$$\overset{\circ}{e}_u := \mathbf{E}[T(u)] = \int_0^\infty {}_t p_u dt = \int_0^1 \overline{F}_u^{-1}(q)dq.$$

Proof.

The first integral follows by integration by parts, and for the second one, and absolutely continuous $\overline{F}$,

$$\begin{aligned}\int_0^\infty \overline{F}_u(x)dx &= \int_0^\infty \overline{F}_u(x)d\overline{F}_u^{-1}(\overline{F}_u(x)) \\ &= \int_1^0 td\overline{F}_u^{-1}(t) = -\int_1^0 \overline{F}_u^{-1}(t)dt \\ &= \int_0^1 \overline{F}_u^{-1}(t)dt,\end{aligned}$$

as required.

Proposition 1.3

For the set up as in the previous proposition, it holds that

$$\textbf{Var}[T(u)] \;=\; 2\int_0^{\infty} t\,{}_tp_u dt - \left(\mathring{e}_u\right)^2.$$

Proof.

Integration by parts. □

Definition 1.1

The median future lifetime of (u) is denoted by $m(u)$, and it can be found by solving:

$$\mathbf{P}(T(u) > m(u)) = \frac{\overline{F}(u + m(u))}{\overline{F}(u)} = 0.5$$

for $m(u)$.

Definition 1.2

The mode for (u) can be obtained as:

$$arg \max_{t \in \mathbf{R}_+} {}_t p_x \mu(x+t).$$

Proposition 1.4

Let $K(u) \backsim F_u$. Then

$$e_u := \mathbf{E}[K(u)] = \sum_{k=1}^{\infty} {}_k p_u.$$

To prove the latter identity, first define the forward difference operator by:

$$\Delta f(x) = f(x+1) - f(x).$$

Remember.

And second recall the summation by parts formula:

$$\sum_{k=m}^{n} f(k)\Delta g(k) = [f(k)g(k)]\,|_m^{n+1} - \sum_{k=m}^{n} g(k+1)\Delta f(k).$$

Proof of Proposition 1.4.

Then:

$$\begin{aligned} e_u &= \sum_{k=0}^{\infty} k\mathbf{P}(K(u)=k) \\ &= \sum_{k=0}^{\infty} k\left(\mathbf{P}(K(u)\le k) - \mathbf{P}(K(u)\le k-1)\right) \\ &= \sum_{k=0}^{\infty} k\left({}_{k+1}q_u - {}_{k}q_u\right). \end{aligned}$$

Proof.

Further

$$\begin{aligned} e_u &= \sum_{k=0}^{\infty} k\Delta_k q_u = \sum_{k=0}^{\infty} k\Delta(1 - {}_k p_u) \\ &= -\sum_{k=0}^{\infty} k\Delta_k p_u. \end{aligned}$$

And hence, using the summation by parts:

$$\begin{aligned} e_u &= -\left(k\,{}_k p_u\big|_0^{\infty} - \sum_{k=0}^{\infty} {}_{k+1}p_u \Delta k \right) \\ &= \sum_{k=0}^{\infty} {}_{k+1}p_u = \sum_{k=1}^{\infty} {}_k p_u, \end{aligned}$$

bearing in mind that e_u is finite, that completes the proof. □

Proposition 1.5

Show that under the UDD assumption, we have that:

$$\overset{\circ}{e}_u = e_u + 0.5.$$

Proof.

$$\begin{aligned}
\overset{\circ}{e}_u &= \frac{1}{l_u}\int_0^\infty l_{u+t}dt = \frac{1}{l_u}\sum_{k=0}^{\infty}\int_0^1 l_{u+k+t}dt \\
&\overset{UDD}{=} \frac{1}{l_u}\sum_{k=0}^{\infty}\int_0^1 ((1-t)l_{u+k} + tl_{u+k+1})\,dt \\
&= \frac{1}{l_u}\sum_{k=0}^{\infty}\left(\frac{1}{2}l_{u+k} + \frac{1}{2}l_{u+k+1}\right) \\
&= \frac{1}{l_u}\left(\frac{1}{2}l_u + \sum^{\infty} l_{u+k}\right) = \frac{1}{2} + e_u
\end{aligned}$$

Important note.

The fact that the UDD holds for (x) does not necesserily imply same on (u).

Proof.

To UDD holds on (u) iff for $k = 0, 1, 2, \ldots$ and $t \in [0,\ 1)$,

$$
\begin{aligned}
& {}_{k+t}p_u = (1-t)\,{}_kp_u + t\,{}_{k+1}p_u \\
\Leftrightarrow\quad & {}_kp_u\,{}_tp_{u+k} = (1-t)\,{}_kp_u + t\,{}_kp_u p_{u+k} \\
\Leftrightarrow\quad & {}_tp_{u+k} = 1 - {}_tq_{u+k} = 1 - t(1-p_{u+k}) = 1 - tq_{u+k} \\
\Leftrightarrow\quad & {}_tq_{u+k} = tq_{u+k}
\end{aligned}
$$

that is true under the UDD. Set $(u) := (x : \overline{n}|)$, then if the UDD holds for (u), it must be

$$
{}_{k+t}p_{x:\overline{n}|} \overset{UDD}{=} (1-t)\,{}_kp_{x:\overline{n}|} + t\,{}_{k+1}p_{x:\overline{n}|},
$$

which is not true, for $k = n-1$, we have $0 \neq (1-t)\,{}_{n-1}p_x$. □

Important.

Note that Proposition 1.5 relates between the fully continuous and fully discrete total expectancy of life. For instance, it connects between the expectation of the r.v.'s $\min(T(x), T(\overline{n|}))$ and

$$\lfloor \min(T(x), T(\overline{n|})) \rfloor = \min\left(\lfloor T(x) \rfloor, \lfloor T(\overline{n|}) \rfloor\right) = \min(K(x), n-1).$$

Importantly, the expectation of the latter r.v. is

$$\mathbf{E}[\min(K(x), n-1)] = \sum_{k=0}^{n-2} k\, {}_{k|}q_x + (n-1)\, {}_{n-1}p_x = \sum_{k=1}^{n-1} {}_k p_x .$$

Proposition 1.6

We have that

$$\mathbf{Var}[K(u)] = \sum_{k=1}^{\infty}(2k-1)\,{}_kp_u - e_u^2.$$

Proof.

We have that

$$\begin{aligned}\mathbf{E}[K(u)^2] &= \sum_{k=0}^{\infty} k^2\,{}_kp_u q_{u+k} \\ &= -\sum_{k=0}^{\infty} k^2 \Delta({}_kp_u) \\ &= -k^2\,{}_kp_u\big|_0^{\infty} + \sum_{k=0}^{\infty} {}_{k+1}p_u \Delta k^2\end{aligned}$$

Proof.

$$\mathbf{E}[K(u)^2] = \sum_{k=0}^{\infty}(2k+1)\,{}_{k+1}p_u = \sum_{k=1}^{\infty}(2k-1)\,{}_{k}p_u,$$

which completes the proof. □

Definition 1.3

The function T_u is defined as:

$$T_u := \int_0^\infty l_{u+t} dt = \int_u^\infty l_y dy,$$

and it represents the expected total future lifetime of a group of l_u individuals all aged u.

To grab an intuition, recall that:

$$\mathring{e}_u = \int_0^\infty {}_t p_u dt = \int_0^\infty \frac{l_{u+t}}{l_u} dt.$$

Hence, the following motivates the definition

$$T_u = l_u \mathring{e}_u.$$

Definition 1.4

The function L_u is defined as:

$$L_u := \int_0^1 l_{u+t} dt = \int_u^{u+1} l_y dy,$$

and it denotes the total expected time lived between ages u and $u + 1$ by l_u lives aged u.

To grab an intuition, note that

$$L_u = T_u - T_{u+1}.$$

Definition 1.5

The central rate of mortality is defined as:

$$m_u := \frac{\int_0^1 l_{u+t}\mu(u+t)dt}{\int_0^1 l_{u+t}dt} = \frac{d_u}{L_u},$$

which is the weighted average force of mortality experienced between ages u and $u+1$ with weight l_{u+t} at age $u+t$.

To get the right most side, divide both numerator and denumerator by l_u, and recall that

$$\int_0^1 {}_tp_u\mu(u+t)dt = q_u,$$

and

$$\int_0^1 tp_u = L_u.$$

Proposition 1.7

The following recursion holds for $f(u)$ being e_u and $\mathring{e}_u$:

$$f(u) = c(u) + d(u)f(u+1) \text{ and } f(u+1) = -\frac{c(u)}{d(u)} + \frac{1}{d(u)}f(u).$$

Proof.

Note first that

$$e_u = \sum_{k=1}^{\infty} {}_kp_u = p_u + \sum_{k=2}^{\infty} {}_kp_u = p_u + p_u \sum_{k=1}^{\infty} {}_kp_{u+1} = p_u + p_u e_{u+1},$$

thus $c(u) = d(u) = p_u,\ u(\infty) = 0$. In the case of $\mathring{e}_u$, and in the fashion similar to the above one, we have that

$$\mathring{e}_u = \int_0^{\infty} {}_tp_u dt = \int_0^1 {}_tp_u dt + \int_1^{\infty} {}_tp_u dt$$

Proof.

$$\mathring{e}_u = \int_0^1 {}_tp_u dt + p_u \int_0^\infty {}_tp_{u+1} dt = \int_0^1 {}_tp_u dt + p_u \mathring{e}_{u+1},$$

and thus

$$c(u) = \int_0^1 {}_tp_u dt \text{ and } d(u) = p_u.$$

□

Remark.

If in addition, we assume linear l_{u+t} for $t \in [0,\ 1]$, then we approximate $c(u)$ above as

$$c(u) \overset{UDD}{=} \int_0^1 (1 - tq_u) dt = 1 - \frac{1}{2} q_u = 1 - \frac{1}{2}(1 - p_u) = \frac{1}{2}(1 + p_u).$$