Life status

Definition 1.1 (Random variable)

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability triple where Ω is sample space, $\mathcal{F}$ is sigma algebra over Ω, and $\mathbf{P}$ is probability measure. Random variable, X, is a map $X : \Omega \to \mathcal{A} \subseteq \mathbf{R}$.

Remember, random variables are in fact functions, i.e., it is more appropriate to write $X(\omega)$ with $\omega \in \Omega$. However, we often (from now on) omit (ω) to make our writing simpler.

Example 1.1 ($\mathcal{A}$ will be sometimes finite in this course)

Let $\Omega = \{sick, healthy, very\ healthy\}$, and define an r.v. as:

$$X(\omega) = \begin{cases} 20, & \omega = \text{sick} \\ 60, & \omega = \text{healthy} \\ 100, & \omega = \text{very healthy} \end{cases} .$$

Example 1.2 ($\mathcal{A}$ does not have to be finite)

Imagine an experiment in which we throw a needle to pick up a number in the interval $(0, 123]$. In such a case, ω denotes an outcome of the experiment, i.e., $\omega \in \Omega = \{(0, 123]\}$. We can then define an r.v. $Y(\omega) = \omega$, with the domain and range being $\mathcal{A} = \{(0, 123]\}$.

Note.

The example above can be easily reformulated to make Y describe a future life time of (x) (= person age x). Note that the maximal future life time of such a person cannot exceed 123.

Definition 1.2 (Continuous future life time of a new born)

Let $T(0) : \Omega \rightarrow \mathbf{R}_+ := (0, \infty)$ denote the r.v. describing the future life time of a new born child.

Definition 1.3 (Continuous future life time of an adult)

Let $x > 0$, then $T(x) = (T(0) - x|\ T(0) > x) : \Omega_x \to \mathbf{R}_+$ denotes an r.v. describing the future life time of a person age x.

Definition 1.4 (Curtate future life time of a new born)

Let $K(0) : \Omega \to \mathbf{Z}_+ := \{0,\ 1,\ \ldots\}$ denote an r.v. describing the curtate future life time of a new born child. We have that $K(0) := \lfloor T(0) \rfloor$.

Definition 1.5 (Curtate future life time of an adult)

Let $x \in \mathbf{Z}_+$, then $K(x) := (K(0) - x|K(0) \geq x) : \Omega_x \to \mathbf{Z}_+$ denotes an r.v. describing the curtate future life time of an adult age x. We again have that $K(x) := \lfloor T(x) \rfloor$.

Note, (x) and (0) above are not related to the concept of sample spaces. These are special cases of what we shall call ‘life status’.

Definition 1.6 (Life status)

A life status (u) is an artificially constructed life form for which the notion of life and death can be well defined.

Example 1.3

The life status $(u) := (0)$ dies when the new born child dies.

Example 1.4

The status $(u) := (x)$ dies when the person age x dies.

Definition 1.7 ('Angle n')

For $n > 0$, we have that $(u) := (\overline{n}|)$ is a life status that survives n time units (alive on $[0,\ n)$) and dies thereafter (dead on $[n,\ \infty)$).

Example 1.5 (Simple joint life status)

The status $(u) := (x : \overline{n}|)$ dies when (x) dies if the death occurs before time n, and it is always dead if otherwise.

Definition 1.8 (Joint life status)

The life status $(x : y)$ is called the joint life status, and it dies on the first death of either (x) or (y).

Note

If payments of, say, an annuity stop when $(x : y)$ dies, then the longer living person is not get paid.

Definition 1.9 (Last survivor life status)

The life status $(\overline{x : y})$ dies on the last death of either (x) or (y).

Note.

- What if the lives of (x) and (y) are dependent?
- The notions above can be extended to augment more than just two lives.

Definition 1.10 (Probability of death)

The c.d.f. of a general life status (u) is given by

$$\mathbf{P}[T(u) \leq t] = F_{T(u)}(t) \text{ or } \mathbf{P}[K(u) \leq k] = F_{K(u)}(k)$$

and it provides the probability that (u) dies during the interval $(0,\ t]$ or $[0,\ k]$.

The c.d.f. F is a map from $R \subseteq \mathbf{R} := (-\infty, \infty)$ to $[0,\ 1]$, with R being the range of the r.v. of interest, such that

- F is non-decreasing, i.e., for $x_1,\ x_2$ in R, and $x_1 \leq x_2$, we have that $F(x_1) \leq F(x_2)$.
- F is right continuous, i.e., $F(x_0+) = \lim_{x \downarrow x_0} F(x) = F(x_0)$.
- $F(+\infty) = 1$,
- $F_{T(u)}(0) = 0$, and $F_{K(u)}(0-) = 0$,
- $F(x_0) = 1 - \overline{F}(x_0)$, for $x_0 \in R$.

Definition 1.11 (Probability to survive)

The d.d.f. (also the survival function) of a general life status (u) is given by

$$\mathbf{P}[T(u) > t] = \overline{F}_{T(u)}(t) \text{ or } \mathbf{P}[K(u) > k] = \overline{F}_{K(u)}(k)$$

and it provides the probability that (u) survives more than t time units.

The d.d.f. $\overline{F}$ is a map from $R \subseteq \mathbf{R} := (-\infty, \infty)$ to $[0,\ 1]$, with R being the range of the r.v. of interest, such that

- $\overline{F}$ is non-increasing, i.e., for $X_1,\ x_2$ in R and $x_1 \leq x_2$, we have that $\overline{F}(x_1) \geq \overline{F}(x_2)$.
- $\overline{F}$ is right continuous, i.e., $\overline{F}(x_0+) = \lim_{x \downarrow x_0} \overline{F}(x) = \overline{F}(x_0)$.
- $\overline{F}(+\infty) = 0$
- $\overline{F}_{T(u)}(0) = 1$, and $\overline{F}_{K(u)}(0-) = 1$,
- $\overline{F}(x_0) = 1 - F(x_0)$, for $x_0 \in R$.

Definition 1.12

We shall denote by

$$ {}_tp_u := \mathbf{P}[T(u) \geq t] = \overline{F}_T(t-) \text{ and } {}_tp := \mathbf{P}[T(0) \geq t] = \overline{F}(t-) $$

the probability that (u) (and (0)) survives at least t time units, and by

$$ {}_tq_u = \mathbf{P}[T(u) < t] = F_T(t-) \text{ and } {}_tq := \mathbf{P}[T(0) < t] = F(t-) $$

the probability that (u) dies before it attains the t-th time unit.

Note.

The events $\{T(u) \geq t\}$ and $\{T(u) < t\}$ are disjoint and complement each other. Thus $1 =_t p_u +_t q_u$, for any fixed $t \geq 0$ and a general life status (u). Same holds for the curtate counterpart.

Note.

Note that ${}_tp_u$ and ${}_tq_u$ are not, respectively, the d.d.f and the c.d.f., however for continuous r.v.'s, we readily have that

$$F(t) - F(t-) = 0 = \mathbf{P}[T(u) = t], \tag{1}$$

and thus in such a case

$${}_tq_u := \mathbf{P}[T(u) < t] \stackrel{(1)}{=} \mathbf{P}[T(u) \le t] = F(t),$$

and

$${}_tp_u := \mathbf{P}[T(u) \ge t] \stackrel{(1)}{=} \mathbf{P}[T(u) > t] = \overline{F}(t).$$

- How do we define a continuous r.v.?
- What other r.v's do exist out there?

Definition 1.13 (Atoms)

Let F be a c.d.f., then the values of $x \in R \subseteq \mathbf{R}$ are called atoms if

$$F(x) - F(x-) > 0.$$

We shall denote by $\mathcal{D}_F$ the set of such atoms of F, i.e.,

$$\mathcal{D}_F := \{x : F(x) - F(x-) > 0\}.$$

Definition 1.14 (Continuous random variables)

The r.v. is continuous if $\mathcal{D}_F = \emptyset$. Hence

$$\sum_{x \in R} (F(x) - F(x-)) = 0.$$

In this course continuous r.v.'s will be denoted by $T(\cdot)$.

- Can you think of any continuous r.v.'s?

Definition 1.15 (Discrete random variables)

The r.v. is discrete if its range R is countable, and hence

$$\sum_{x \in R} (F(x) - F(x-)) = 1.$$

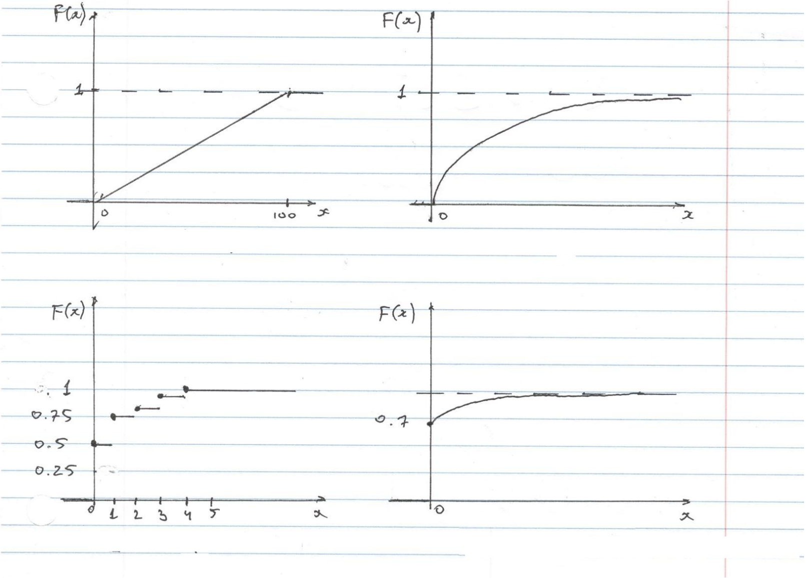
In this course discrete r.v.'s will be denoted by $K(\cdot)$.

Definition 1.16 (Mixed random variables)

The r.v. is mixed if its c.d.f. is such that

$$0 < \sum_{x \in R} (F(x) - F(x-)) < 1.$$

- We shall see one more type of random variables in this course. Any guess?


F(x)
1
0
100
x
F(x)
1
0
x
F(x)
1
0.75
0.5
0.25
0
1
2
3
4
5
x
F(x)
0.7
0
x

Definition 1.17 (Absolutely continuous random variables)

The r.v. is absolutely continuous if there exists a function (density) $f(x) > 0,\ x \in R$ such that, for $a < b$, we have that

$$F(b) - F(a) = \int_a^b f(x)dx.$$

If f exists, then it is called the probability density function (p.d.f.).

Useful absolutely continuous r.v.'s to remember:

- Exponential: $T(0) \backsim Exp(\lambda), \lambda > 0$. The p.d.f. is $f(x) = \lambda e^{-\lambda x}\mathbf{1}\{x > 0\}$. The c.d.f. is $F(x) = 1 - e^{-\lambda x}$.
- Gamma: $T(0) \backsim Ga(\gamma, \alpha), \gamma, \alpha > 0$. The p.d.f. is $f(x) = e^{-\alpha x}x^{\gamma-1}\alpha^{\gamma}(\Gamma(\gamma))^{-1}\mathbf{1}\{x > 0\}$. The c.d.f. is...?
- Normal: $N(\mu, \sigma^2)$. The p.d.f. is $f(x) = (\sqrt{2\pi\sigma^2})^{-1} exp\left\{-0.5(x-\mu)^2/\sigma^2\right\}\mathbf{1}\{x \in \mathbf{R}\}$

Definition 1.18 (Force of mortality.)

For an absolutely continuous r.v. $T(0)$ with the c.d.f. F, d.d.f. $\overline{F}$ and p.d.f. f, and $t > 0$, the force of mortality is

$$\mu(t) := \frac{f_{T(0}(t)}{\overline{F}_{T(0)}(t)}.$$

Motivation for $\mu(\cdot)$.

For $T(0)$ as above, we have that

$$\begin{aligned}
\mu_{T(0)}(t) &= \frac{F'(t)}{\overline{F}(t)} = \frac{1}{\overline{F}(t)} \lim_{h \downarrow 0} \frac{F(t+h) - F(t)}{h} \\
&= \frac{1}{\overline{F}(t)} \lim_{h \downarrow 0} \frac{\mathbf{P}[t < T(0) \leq t+h]}{h} \\
&= \lim_{h \downarrow 0} \frac{\mathbf{P}[t < T(0) \leq t+h |\ T(0) > t]}{h}.
\end{aligned}$$

Remember

For each age $t > 0$, $\mu(t)$ gives the value of the conditional density function of $T(0)$ at exact age t given survival to that age.

At home

Show that the force of mortality of $T(u)$, if exists, is given by

$$\mu_{T(u)}(t) = \mu(u + t).$$

Remember

In actuarial notation, we have that

$$\mu_{T(u)}(t) = \frac{1}{{}_tp_u} \frac{d}{dt} {}_tq_u = -\frac{1}{{}_tp_u} \frac{d}{dt} {}_tp_u = \frac{1}{{}_{t+u}p} \frac{d}{dt} {}_{t+u}q.$$

- It is possible to find a relation between the force of mortality and the probability of survival/death.

Proposition 1.1

Let $T(u)$ have the force of mortality $\mu(u+t)$. Then

$$ {}_tp_u = \mathbf{P}[T(u) \geq t] = \overline{F}_{T(u)}(t) = \exp\left\{-\int_0^t \mu(u+s)ds\right\}. $$

Proof.

Start with the definition$\mu(u+s) = -\overline{F}'(u+s)/\overline{F}(u+s)$, for a fixed u, and rewrite it as

$$ \mu(u+s) = -\frac{d}{ds}\ln\overline{F}(u+s). $$

Integrate both sides

$$ -\int_0^t \mu(u+s)ds = \int_0^t \frac{d}{ds}\ln\overline{F}(u+s) = \ln\frac{\overline{F}(u+t)}{\overline{F}(u)}, $$

Cont.

and notice that

$$\begin{aligned} {}_tp_u &= \mathbf{P}[T(u) \geq t] = \mathbf{P}[T(u) > t] \\ &= \mathbf{P}[T(0) > u + t |\ T(0) > u] = \frac{\mathbf{P}[T(0) > u + t]}{\mathbf{P}[T(0) > u]} \\ &= \frac{\overline{F}(u+t)}{\overline{F}(u)} = \frac{{}_{u+t}p}{{}_up}. \end{aligned}$$

This

$$-\int_0^t \mu(u+s)ds = \ln {}_tp_u,$$

which leads to

$${}_tp_u = \exp\left\{-\int_0^t \mu(u+s)ds\right\},$$

and hence completes the proof.

At home.

In the proof of Proposition 1.1, we showed that

$$ {}_t p_u = {}_{u+t} p / {}_u p $$

for any positive u and t and a continuous r.v. $T(u)$. Show that the continuity is not necessary, and the statement is still true.

Note.

From Proposition 1.1, we have that (change of variables),

$$ {}_t p_u = \exp\left\{ -\int_u^{u+t} \mu(s) ds \right\}, $$

and also (put $u = 0$),

$$ {}_t p = \exp\left\{ -\int_0^t \mu(s) ds \right\}, $$

Proposition 1.2

Let $T(u)$ be an absolutely continuous r.v., then its p.d.f. is, for positive u and t,

$$f_{T(u)}(t) = {}_t p_u \mu(u+t).$$

Proof.

Use Proposition 1.1 and have that

$$-\frac{d}{dt} {}_t p_u = \exp\left\{ -\int_u^{u+t} \mu(s) ds \right\} \mu(u+t) = {}_t p_u \mu(u+t).$$

which completes the proof. □

Note.

Put $u = 0$, and the p.d.f. of $T(0)$ becomes

$$f(t) = {}_t p \mu(t).$$

Proposition 1.3 (Property of μ.)

If μ is a legitimate force of mortality, then we have that

$$\lim_{t\uparrow\infty}\int_u^{u+t}\mu(s)ds=\infty.$$

Proof.

Note that $\lim_{t\uparrow\infty} {}_tp_u = 0$ and hence $\lim_{t\uparrow\infty} \ln {}_tp_u = -\infty$. Then use Proposition 1.1 and have that

$$\lim_{t\uparrow\infty}\int_u^{u+t}\mu(s)ds=-\lim_{t\uparrow\infty}\ln {}_tp_u=\infty,$$

which completes the proof. □

- Proposition 1.3 means that not every function can serve as a force of mortality.

Remember.

For a function to be a legitimate force of mortality, we must have that

- $\mu(u+t) \geq 0$ - conditional p.d.f., and
- $\lim_{t\uparrow\infty} \int_u^{u+t} \mu(s)ds = \infty$ - the accumulated force of mortality eventually becomes infinitely strong and kills.

At home.

We said μ has an interpretation of a conditional density, why then do we have Proposition 1.3? We know that a density integrates to one.

- Is it very practical to consider continuous r.v.'s describing future life times?

Proposition 1.4

Let $K(u)$ be a curtate r.v. representing the future life time of a life status (u). The probability mass function (p.m.f.) of $K(u)$ is given by

$$\mathbf{P}[K(u) = k] =_k p_u \cdot q_{u+k}.$$

Proof.

Remember that $K(u)$ is the highest integer in $T(u)$. We have that, for $k = 0, 1, \ldots,$

$$\begin{aligned} & \mathbf{P}[K(u) = k] = \mathbf{P}[k \leq T(u) < k+1] = \mathbf{P}[k < T(u) \leq k+1] \\ = \; & F_{T(u)}(k+1) - F_{T(u)}(k) =_{k+1} q_u -_k q_u =_k p_u \cdot q_{u+k} \end{aligned}$$

Also $\sum_{k=0}^{\infty} {}_k p_u \cdot q_{u+k} =_\infty q_u$, and the p.m.f. is a legitimate one. This completes the proof. □

Remember.

We shall denote the probability $\mathbf{P}[K(u) = k]$ by ${}_{k|}q_u$, that is a deferred probability of death during the time unit following the k-th one. In a similar fashion, the deferred probability to die in n time units during the consequent m time units is ${}_{n|m}q_u$.

- Remember that for a discrete r.v. $K(u)$, the probability $\mathbf{P}[K(u) \leq k]$ is not equal to ${}_k q_u$. So what is the c.d.f. of $K(u)$?

Proposition 1.5

The c.d.f. of $K(u)$ is given by, for $k = 0, 1, \ldots,$

$$F_{K(u)}(k) =_{k+1} q_u.$$

Proof.

$$F_{K(u)}(k) = \sum_{l=0}^{k} \mathbf{P}[K(u) = l] = \sum_{l=0}^{k} {}_{l|}q_u$$

$$= \sum_{l=0}^{k} {}_{l+1}q_u - {}_{l}q_u = {}_{k+1}q_u - {}_{0}q_u = {}_{k+1}q_u,$$

which completes the proof. □

Remember.

We assumed that the observation of survival at age u yields the same conditional distribution of survival as the hypothesis that (0) survived to age u, i.e.,
$\mathbf{P}[T(u) > t] = \mathbf{P}[T(0) > u + t | \ T(0) > u]$. Natural?

Simple life status (0).

At home.

Show that, for positive $n,\ m,\ u$,

$$_{n|m}q_u =_n p_u \cdot_m q_{u+n}.$$

Explain.

Example 1.6 (Simple life status for a new born.)

Let $(u) = (0)$, i.e., we deal with a life status that dies when a new born child dies. In such a case, $K(0)$ describes the future life time of the child in discrete time units (often years), and $T(0)$ describes it continuously. For the former, we have that the p.m.f. is $_{k|}q$ and the c.d.f. is $_{k+1}q$. Also, for the latter, we have that the p.d.f. is $_tp\mu(t)$, the c.d.f. is $_tq$ and the force of mortality is $\mu(t)$. Here $k = 0, 1, \ldots$ and $t > 0$.

Simple life status (x).

Example 1.7 (Simple life status for a person age x.)

Let $(u) = (x)$, i.e., we deal with a life status that dies when a person age x dies. In such a case, $K(x)$ describes the future life time of that person in discrete time units (often years), and $T(x)$ describes it continuously. For the former, we have that the p.m.f. is ${}_{k|}q_x$ and the c.d.f. is ${}_{k+1}q_x$. Also, for the latter, we have that the p.d.f. is ${}_tp_x\mu(x+t)$, the c.d.f. is ${}_tq_x$ and the force of mortality is $\mu(x+t)$. Here $k = 0, 1, \ldots$, $x > 0$ and $t > 0$.

Simple life status ($\overline{n|}$).

Example 1.8

Let $(u) = (\overline{n|})$, i.e., we specialize our general life status to the one that is alive before time n and is dead thereafter. The future life time is therefore described by the r.v. $T(\overline{n|})$, such that

$$ {}_t q_{\overline{n|}} = \begin{cases} 0, & t < n \\ 1, & t \geq n \end{cases} \Leftrightarrow {}_t\, p_{\overline{n|}} = \begin{cases} 1, & t < n \\ 0, & t \geq n \end{cases} . $$

$K(\overline{n|}) := \lfloor T(\overline{n|}) \rfloor$, such that

$$ \mathbf{P}[K(\overline{n|}) = k] = \begin{cases} 1, & k = n-1 \\ 0, & k \neq n-1 \end{cases} , $$

$$ {}_k p_{\overline{n|}} = \begin{cases} 1, & k \leq n-1 \\ 0, & k > n-1 \end{cases} = \begin{matrix} 1, & k < n \\ 0, & k \geq n \end{matrix} $$

At home.

If one dollar is payed upon death of $(\overline{n|})$, what is the a.p.v. of the payment given an interest rate i?

A simple joint life status $(x : \overline{n|})$.

Example 1.9 (A simple joint life status $(x : \overline{n|})$.)

Let $(u) = (x : \overline{n|})$, with the corresponding r.v. describing the future life time being $T(x : \overline{n|})$. The life status is a joint life status and it thus dies on the first death of either (x) or $(\overline{n|})$. Then

$$ {}_t p_{x:\overline{n|}} = \mathbf{P}[T(x : \overline{n|}) \geq t] = \begin{cases} \mathbf{P}[T(x) \geq t] = {}_t\, p_x, & t < n \\ 0, & t \geq n \end{cases} $$

The r.v. is a mixed one with an atom (jump) at $t = n$. Also

$$ {}_k p_{x:\overline{n|}} = \mathbf{P}[K(x : \overline{n|}) \geq k] = \begin{cases} \mathbf{P}[K(x) \geq k] = {}_k\, p_x, & k < n \\ 0, & k \geq n \end{cases} $$

At home.

Show that the expectation of $T(x:\overline{n}|)$ is given by

$$\mathbf{E}[T(x:\overline{n}|)] = \int_0^n t\,{}_tp_x \cdot \mu(x+t)dt + n\,{}_np_x.$$

Also, check that

$$\mathbf{E}[K(x:\overline{n}|)] = \sum_{k=0}^{n-1} k\,{}_{k|}q_x + (n-1)\,{}_np_x.$$

Joint life status.

Example 1.10 (A joint life status $(x:y)$.)

Let $(u) = (x:y)$, with the corresponding r.v. describing the future life time being $T(x:y)$. The life status is a joint life status and it thus dies on the first death of either (x) or (y). Let $T(x)$ be independent of $T(y)$. Then

$$ {}_t p_{x:y} = \mathbf{P}[T(x:y) \geq t] = {}_t p_x \cdot {}_t p_y . $$

Ordered deaths.

Example 1.11 (A life status $(\overset{1}{x} : \overline{n|})$.)

Let $(u) = (\overset{1}{x} : \overline{n|})$ be a life status that dies if (x) dies first (on $[0,\ n)$), and it never dies thereafter (on $[n,\ \infty)$) if (x) is then alive. In such a case

$$ {}_t p_{\overset{1}{x}:\overline{n|}} = \mathbf{P}[T(\overset{1}{x} : \overline{n|}) \geq t] = \begin{cases} {}_t p_x, & t < n \\ {}_n p_x, & t \geq n \end{cases} . $$

Also

$$ {}_k p_{\overset{1}{x}:\overline{n|}} = \mathbf{P}[K(\overset{1}{x} : \overline{n|}) \geq k] = \begin{cases} {}_k p_x, & k < n \\ {}_n p_x, & k \geq n \end{cases} . $$

At home.

Show that the expectation of $T(\overset{1}{x} : \overline{n}|)$ is given by

$$\mathbf{E}[T(\overset{1}{x} : \overline{n}|)] = \infty.$$

Also, check that

$$\mathbf{E}[K(\overset{1}{x} : \overline{n}|)] = \infty.$$

Example 1.12 (A life status $(x : \overset{1}{\overline{n|}})$.)

Let $(u) = (x : \overset{1}{\overline{n|}})$ be a life status that dies if $(\overline{n|})$ dies first. In such a case

$$
{}_tp_{x:\overset{1}{\overline{n|}}} = \mathbf{P}[T(x : \overset{1}{\overline{n|}}) \geq t] = \begin{cases} 1, & t < n \\ {}_nq_x, & t \geq n \end{cases} .
$$

Also

$$
{}_kp_{x:\overset{1}{\overline{n|}}} = \mathbf{P}[K(x : \overset{1}{\overline{n|}}) \geq k] = \begin{cases} 1, & k < n \\ {}_nq_x, & k \geq n \end{cases} .
$$

At home.

Show that the expectations with respect to the last two p functions are infinite.