Mathematics of Life Contingencies. Math 3280 3.00 F

Instructor: Edward Furman

Homework 3—solution

Unless otherwise indicated, all lives in the following questions are subject to the same law of mortality and their times until death are independent random variables.

1. If $s(x) = 1 - \frac{x}{100}$, $0 \leq x \leq 100$, calculate:

   a. $\mu(x)$

   b. $F_X(x)$

   c. $f_X(x)$

   d. $\Pr(10 < X < 40)$.

   solution

   This is De Moivre's Law with w=100, hence,

   a) $\mu(x) = \frac{1}{w-x} = \frac{1}{100-x}$,

   b) $F_X(x) = \frac{x}{100}$,

   c) $f_X(x) = \frac{1}{100}$,

   d) $\Pr(10 < X < 40) = \frac{40-10}{w} = 0.3$.

2. Given the survival function of question 1, determine the survival function, force of mortality, and p.d.f of the future lifetime of (40).

   solution

   With De Moivre's Law, future lifetime of (x) is uniformly distributed on [0, w-x]. In this case, this means a uniform distribution on [0, 60]. Therefore,

$$S_{T(40)}(t) = \frac{w-x-t}{w-x} = \frac{60-t}{60}$$

$$\mu(40+t) = \frac{1}{60-t}$$

$$f_{T(40)}(t) = \frac{1}{60}$$

3. If $\mu(x) = 0.001$ for $20 \leq x \leq 25$, evaluate ${}_{2|2}q_{20}$.

solution

The force of mortality is constant and equal to 0.001 throughout the age interval under consideration, therefore:

$${}_{2|2}q_{20} = {}_2 p_{20} \cdot {}_2 q_{22} = (e^{-0.001})^2(1 - (e^{-0.001})^2) = 0.00199401$$

4. Show that

$$\frac{d}{dx}\ {}_tp_x \;=\; {}_tp_x\ [\mu(x) - \mu(x+t)]$$

solution

$\frac{d}{dx}\ {}_tp_x = \frac{d}{dx}\ \frac{l_{x+t}}{l_x} \;=\; \frac{l_x(-l_{x+t}\mu(x+t)) - l_{x+t}(-l_x\mu(x))}{(l_x)^2} = {}_t p_x\ [\mu(x) - \mu(x+t)]$

5. If $\mu(x+t) =$ t, $t \geq 0$, calculate ${}_tp_x\ \mu(x+t)$.

solution

${}_tp_x = e^{-\int_0^t \mu(x+r)dr} = e^{-\int_0^t r dr} = e^{-\frac{1}{2}t^2}$. Therefore, $f_T(t) = te^{-\frac{1}{2}t^2}$.

6. You are given that

$$s(x) = (\frac{100}{100+x})^2$$

Calculate ${}_{5|}q_{40}$.

solution

$$\begin{aligned}
{}_{5|}q_{40} &= \frac{F(46) - F(45)}{s(40)} \\
&= \frac{s(45) - s(46)}{s(40)} \\
&= \frac{(\frac{100}{145})^2 - (\frac{100}{146})^2}{(\frac{100}{140})^2} \\
&= 0.012726
\end{aligned}$$

7. You are given $\mu_x = \frac{a}{w-x}$, prove that

$$_tp_x = (\frac{w-x-t}{w-x})^a$$

solution

$$_tp_x = e^{-\int_0^t \mu(x+r)dr} = e^{-\int_0^t \frac{a}{w-x-r}}dr = e^{a \cdot ln(w-x-r)|_0^t} = e^{ln(\frac{w-x-t}{w-x})^a} = (\frac{w-x-t}{w-x})^a$$

8. You are given that the force of mortality is

$$\mu_x = \frac{0.5}{100-x}$$

Calculate the probability that (36) survives to age 75.

solution

We recognize this as generalized deMoivre with a=0.5 and w=100, so the answer is

$$_{39}p_{36} = (\frac{100-36-39}{100-36})^{0.5} = 0.625$$

9. You are given:

1) $\hat{\mu}_{x+t} = \mu_{x+t} - k,\ 0 \leq t \leq 1$

2) $\hat{q}_x = 0$ where $\hat{q}_x$ is based on the force of mortality $\hat{\mu}_{x+t}$

Determine k.

solution

$$1 = \hat{p}_x = exp(-\int_0^1 (\mu_t - k)dt)$$
$$= p_x \ \ exp(-\int_0^1 (-k)dt)$$
$$= p_x e^k$$

So $e^k = \frac{1}{p_x}$, and k=-ln p_x.

GOOD LUCK!