Mathmatics of Life Contingencies. Math 3280 3.00 F

Instructor: Edward Furman

Homework 2——Solution

Unless otherwise indicated, all lives in the following questions are subject to the same law of mortality and their times until death are independent random variables.

1. Confirm that the following can serve as survival function. Show the corresponding $\mu(x)$, $f_X(x)$, and $F_X(x)$.

$$S(x) = e^{-x^3/12}, x \geq 0.$$

Solution:

It is continuous, non-increasing, and s(0)=1, while

$$\lim_{x\to\infty} e^{-\frac{x^3}{12}} = 0$$

Furthermore,

$$F_X(x) = 1 - e^{-x^3/12}$$

Also,

$$f_X(x) = F'_X(x) = \frac{1}{4}x^2 e^{-x^3/12}, \quad x \geq 0.$$

And

$$\mu(x) = \frac{f_X(x)}{s(x)} = \frac{1}{4}x^2, \quad x \geq 0.$$

2. If the random variable T(x) has d.f. given by

$$F_{T(x)}(t) = \begin{cases} \frac{t}{(100-x)} & 0 \leq t < 100 - x \\ 1 & t \geq 100 - x \end{cases}$$

calculate:

a. $\mathbf{Var}[T(x)]$.

b. median [T(x)].

Solution:

We have:

$$f_{T(x)}(t) = \frac{d}{dt}\frac{t}{100-x} = \frac{1}{100-x}$$

This means that the density of T(x) is constant, and T(x) has uniform distribution on [0, 100 - x]. This gives us immediately:

a. $\mathbf{Var}(T) = \frac{(100-x)^2}{12}$.

b. $\mathbf{Median}(T) = \frac{100-x}{2}$.

3. The joint p.d.f. of T(x) and T(y) is given by

$$f_{T(x)T(y)}(s,t) = \begin{cases} 0.0006(t-s)^2 & 0 < s < 10, 0 < t < 10. \\ 0 & otherwise \end{cases}$$

Find:

a. The joint c.d.f. of T(x) and T(y).

b. The p.d.f. and c.d.f. and $\mu(x+s)$ for the marginal distribution of T(x). Note the symmetry in s and t which implies that T(x) and T(y) are identically distributed.

c. The covariance and correlation coefficient of T(x) and T(y).

Solution:

a.

$$F_{T(x)T(y)}(s,t) = \int_{-\infty}^{s}\int_{-\infty}^{t} f_{T(x)T(y)}(u,v)\ du\ dv$$
$$= \int_{0}^{s}\int_{0}^{t} 0.0006(v-u)^2\ dv\ du$$
$$= 0.00005[s^4 + t^4 - (t-s)^4]$$

$$0 < s \leq 10, \quad 0 < t \leq 10.$$

b. Using the c.d.f. obtained in part(a), we have

$$F_{T(x)T(y)}(s,10) = F_{T(x)}(s) = \begin{cases} 0 & x \leq 0 \\ 0.5 + 0.00005[s^4 - (10-s)^3] & 0 < s \leq 10 \\ 1 & s > 10 \end{cases}$$

And

$$f_{T(x)}(s) = F'_{T(x)}(s) = 0.0002[s^3 + (10-s)^3]. \qquad 0 < s \leq 10$$

And

$$\mu(x+t) = \frac{f_{T(x)}(s)}{1 - F_{T(x)}(s)}$$

$$= \frac{0.0002[s^3 + (10-s)^3]}{0.5 - 0.00005[s^4 - (10-s)^3]} \qquad 0 < s \leq 10$$

c.

$$\mathbf{E}[T(x)] = \int_0^{10} s(0.0002)[s^3 + (10-s)^3]ds = 5 = \mathbf{E}[T(y)]$$

$$\mathbf{E}[T(x)^2] = \int_0^{10} s^2(0.0002)[s^3 + (10-s)^3]ds = 110/3 = \mathbf{E}[T(y)^2]$$

$$\mathbf{Var}[T(x)] = 35/3 = \mathbf{Var}[T(y)]$$

$$\mathbf{E}[T(x)T(y)] = \int_0^{10}\int_0^{10} st(0.0006)(t-s)^2\ ds\ dt = \frac{50}{3}$$

$$\mathbf{Cov}[T(x), T(y)] = \mathbf{E}[T(x)T(y)] - \mathbf{E}[T(x)]\mathbf{E}[T(y)] = -\frac{25}{3}$$

$$\rho_{T(x)T(y)} = \frac{\mathbf{Cov}[T(x), T(y)]}{\sigma_{T(x)}\sigma_{T(x)}} = -\frac{5}{7}$$

4. Display the joint survival function of (T(x), T(y)) where the distribution is defined in Problem 1.

Solution:

$$S_{T(x)T(y)}(s,t) = \int_s^{\infty}\int_t^{\infty} f_{T(x)T(y)}(u,v)\ du\ dv$$

$$= \int_s^\infty \int_t^\infty 0.0006(v-u)^2 \; du \; dv$$

$$= 0.00005[(10-t)^4 + (10-s)^4 - (t-s)^4]$$

5. In terms of the single life probabilities ${}_np_x$ and ${}_n\mathbf{p}_y$, express

a. The probability that (x : y) will survive at least n years.

b. The probability that exactly one of the lives (x) and (y) will survive at least n years.

c. The probability that at least one of the lives (x) and (y) will survive at least n years.

d. The probability that (x : y) will fail within n years.

e. The probability that at least one of the lives will die within n years.

f. The probability that both lives will die within n years.

Solution:

a.

$${}_np_x \cdot {}_np_y$$

b.

$$Pr[T(x) > n \textbf{ and } T(y) \leq n, \textbf{ or } T(y) > n \textbf{ and } T(x) \leq n]$$

$$= {}_n p_x(1 - {}_n p_y) + {}_n p_y(1 - {}_n p_x) = {}_n p_x + {}_n p_y - 2 \cdot {}_n p_x \cdot {}_n p_y.$$

c.

$${}_np_{\overline{xy}} = {}_n p_x + {}_n p_y - {}_n p_x \cdot {}_n p_y$$

d.

$${}_nq_{xy} = 1 - {}_n p_{xy} = 1 - {}_n p_x \cdot {}_n p_y$$

e.

$$1 - {}_n p_x \cdot {}_n p_y$$

f.

$$ {}_nq_x \cdot_n q_y = 1 -_n p_x -_n p_y +_n p_x \cdot_n p_y $$

6. For two lives with lifetime variables S and T, you are given that f(s,t) $= \frac{3t^2}{10^8}$ for $0 < s < 100, 0 < t < 100$.

   Calculate the probability that the joint status (0:0) survives 60 years.

   Solution:

$$ Pr(S > 60, T > 60) = \int_{60}^{100} \int_{60}^{100} \frac{3t^2}{10^8}\, ds\ dt $$
$$ = \frac{120}{10^8} \frac{100^3 - 60^3}{3} = 0.3136 $$

7. Show that the probability that (x) survives at least n years and (y) survives n 1 years may be expressed either as

$$ \frac{{}_n\mathbf{p}_{x:y-1}}{\mathbf{p}_{y-1}} $$

   or as

$$ \mathbf{p}_x \quad {}_{(n-1)}\mathbf{p}_{x+1:y} $$

   Solution:

   We seek ${}_np_x \cdot_{n-1} p_y$, which is $p_x \cdot_{n-1} p_{x+1} \cdot_{n-1} p_y$, or $p_x \cdot_{n-1} p_{(x+1):y}$.

   Alternatively, ${}_np_(y-1) = p_{y-1} \cdot_{n-1} p_y$, so that ${}_{n-1}p_y = \frac{{}_np_{y-1}}{p_{y-1}}$, producing $\frac{{}_np_{x:y-1}}{p_{y-1}}$.

8. If $\overline{A}$ refers to the complement of the event A within the sample space and $Pr(\overline{A}) \neq 0$, the following expresses an identity in probability theory:

$$ Pr(A \cup B) = Pr(A) + Pr(\overline{A})Pr(B|\overline{A}) $$

Rewrite this identity in actuarial notation for the events $A = [T(x) \leq t]$ and $B = [t < T(x) \leq 1]$, $0 < t < 1$.

Solution:

$Pr(A \cup B)$ becomes $Pr[T(x) \leq 1] = q_x$, $Pr(A)$ is ${}_tq_x$, and $Pr(B|\overline{A})$ is ${}_{1-t}q_{x+t}$; hence,

$$q_x =_t q_x +_t p_x \cdot_{1-t} q_{x+t}$$

—End—