Question 1 Let ${}_tp = (1 - t/\omega)^\alpha$ where $0 \le t < w$ and $\alpha > 0$. Derive $\mu_x(t)$ and $\mathring{e}_x$ for positive x and t.

Question 2 Show that, for a positive RV $T(x)$, $x \ge 0$, we have

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt$$

Question 1 Sol

$${}_tp_x = {}_{x+t}p / {}_xp = \frac{\left(1 - \frac{t+x}{w}\right)^\alpha}{\left(1 - \frac{x}{w}\right)^\alpha} = \left(1 - \frac{t}{w-x}\right)^\alpha$$

So

$$\mu_x(t) = \frac{1}{{}_tp_x}(-1)\frac{d}{dt}\,{}_tp_x = \alpha\left(1 - \frac{t}{w-x}\right)^{\alpha-1}\frac{1}{w-x}\left(1 - \frac{t}{w-x}\right)^{-\alpha}$$

$$= \frac{\alpha}{w-x}\,\frac{w-x}{(w-x-t)} = \frac{\alpha}{w-x-t}, \quad t \in (0, w-x)$$

Also

$$\mathring{e}_x = \int_0^{w-x}\left(1 - \frac{t}{w-x}\right)^\alpha dt = \frac{1}{\alpha+1}\int_0^{w-x} d\left(1 - \frac{t}{w-x}\right)^{\alpha+1}(-1)(w-x)$$

$$= \frac{w-x}{\alpha+1}(-1)\left(1 - \frac{t}{w-x}\right)\Bigg|_0^{w-x} = \frac{w-x}{\alpha+1}$$

Question 2 Sol

$$\mathring{e}_x = \mathbb{E}[T(x)] = \mathbb{E}\left[\int_0^{T(x)} dt\right] = \mathbb{E}\left[\int_0^\infty \mathbb{I}\{t \le T(x)\}\, dt\right]$$

$$= \int_0^\infty \mathbb{E}[\mathbb{I}\{T(x) \ge t\}]\, dt = \int_0^\infty {}_tp_x \, dt$$