Definition 0.1 (Life annuity.)

Life annuity is a series of payments made continuously or at equal intervals while a given life survives.

- Unlike in the context of life insurances, we have a series of payments.
- Discrete payments can be made not only at the end of the year, but also at the beginning of the year.

Question.

Given a life status (u) what is the expected future lifetime? If one dollar is payed at the beginning of every year while (u) is alive, what would the random present value of the contract be? What if the payment was made at the end of the year?

Recall that, e.g., $K(u) : \Omega \to \mathcal{N}_u \subseteq [0, \infty]$.

Solution.

- The expected future lifetime is $\mathbf{E}[K(u)]$ full years.
- The random present value if the payments are made at the beginning of each year is $\ddot{a}_{\overline{K(u)+1}|} = 1 + v + \cdots + v^{K(u)+1-1}$.
- Also, the random present value of the payments if they are made at the end of each year is $a_{\overline{K(u)}|} = v + \cdots + v^{K(u)}$.

Once again, recall that we find the price by taking expectation operator (under the identity utility principle).

Net premium for an annuity contract

Fix a life status (u) (be careful with joint life statuses). The net premium for a continuously payable annuity is

$$\mathbf{E}[\overline{a}_{\overline{T(u)}|}] = \int_{\mathcal{R}_u} \overline{a}_{\overline{t}|} {}_t p_u \mu(u+t) dt = -\int_{\mathcal{R}_u} \overline{a}_{\overline{t}|} d\, {}_t p_u := \overline{a}_u.$$

Net premium for an annuity contract. cont.

The net premium for an annuity due payable annually is

$$\mathbf{E}[\ddot{a}_{\overline{K(u)+1}|}] = \sum_{k\in\mathcal{N}_u} \ddot{a}_{\overline{k+1}|k}p_u q_{u+k} = -\sum_{k\in\mathcal{N}_u} \ddot{a}_{\overline{k+1}|}\Delta_k p_u := \ddot{a}_u.$$

Last but not least, the net premium for an annuity certain payable annually is

$$\mathbf{E}[a_{\overline{K(u)}|}] = \sum_{k\in\mathcal{N}_u} a_{\overline{k}|k}p_u q_{u+k} = -\sum_{k\in\mathcal{N}_u} a_{\overline{k}|}\Delta_k p_u := a_u.$$

Example 0.1

Whole life annuity. Let $(u) = (x)$. Then $\mathcal{R}_u = [0, \infty)$ and $\mathcal{N}_u = \{0,\ 1, \ldots\}$. Thus

Cont.

$$\ddot{a}_x = \mathbf{E}[\ddot{a}_{\overline{K(x)+1}|}] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|k|} q_x .$$

Further

$$a_x = \mathbf{E}[a_{\overline{K(x)}|}] = \sum_{k=0}^{\infty} a_{\overline{k}|k|} q_x .$$

And

$$\bar{a}_x = \mathbf{E}[\bar{a}_{\overline{T(x)}|}] = \int_0^{\infty} \bar{a}_{\overline{t}|t} p_x \mu(x+t) dt .$$

Recall that.

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}, \; a_{\overline{n}|} = \frac{1-V^n}{i}, \; \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}, \; n \geq 0.$$

Proposition 0.1

Fix a life status (u), and let i, d, δ be the effective and discount interests and the force of interest, respectively. Then,

$$\bar{a}_u = \frac{1-\bar{A}_u}{\delta}, \quad a_u = \frac{1-(1+i)A_u}{i}, \quad \ddot{a}_u = \frac{1-A_u}{d}.$$

Proof.

We clearly have that

$$\bar{a}_u = \mathbf{E}\left[\bar{a}_{\overline{T(u)}|}\right] = \mathbf{E}\left[\frac{1-v^{T(u)}}{\delta}\right] = \frac{1-\bar{A}_u}{\delta}.$$

The same can be done for $\ddot{a}_u$. For the last one

$$a_u = \mathbf{E}\left[\frac{1-v^{K(u)}}{i}\right] = \mathbf{E}\left[\frac{1-(1+i)v^{K(u)+1}}{i}\right],$$

as required. □

Proposition 0.2

Fix (u). The variance of the random present value can be given by

$$\mathbf{Var}[\overline{a}_{\overline{T(u)}|}] = \frac{{}^2\overline{A}_u - (\overline{A}_u)^2}{\delta^2},$$

and

$$\mathbf{Var}[\ddot{a}_{\overline{K(u)+1}|}] = \frac{{}^2A_u - (A_u)^2}{d^2},$$

as well as

$$\mathbf{Var}[a_{\overline{K(u)}|}] = \frac{(1+i)^2\left({}^2A_u - (A_u)^2\right)}{i^2}.$$

Proof.

The proof is straightforward and similar to the proof of the previous proposition. It is thus omitted. □

Proposition 0.3

Consider a whole life annuity due. Then

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x = \sum_{k=0}^{\infty} {}_k E_x.$$

Proof.

We have that

$$\begin{aligned}
\ddot{a}_x &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|k} p_x q_{x+k} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} v^i \mathbf{P}[K(x) = k] \\
&= \sum_{i=0}^{\infty} \left(\sum_{k=i}^{\infty} \mathbf{P}[K(x) = k] \right) v^i = \sum_{i=0}^{\infty} \mathbf{P}[K(x) \geq i] v^i \\
&= \sum_{i=0}^{\infty} \mathbf{P}[K(x) > i - 1] v^i = \sum_{i=0}^{\infty} {}_i p_x v^i,
\end{aligned}$$

as required. □

Proposition 0.4

Let again $(u) = (x)$. *Then*

$$\ddot{a}_x = 1 + E_x \ddot{a}_{x+1}.$$

Here $\ddot{a}_\infty = 0$.

Remark.

Note that $\ddot{a}_\infty \neq \ddot{a}_{\overline{\infty}|} = 1/d$.

Proof.

We have that

$$\ddot{a}_x = 1 + \sum_{k=0}^{\infty} {}_{k+1}p_x v^{k+1} = 1 + vp_x \sum_{k=0}^{\infty} v^k {}_k p_{x+1},$$

as required. □

Example 0.2 (n year term life annuity.)

Let $\mathcal{R}_u = [0, \infty)$ and $\mathcal{N}_u = \{0, 1, \ldots, \infty\}$. And let $(u) = (x : \overline{n}|)$. In the discrete case, $K(x : \overline{n}|)$ is $K(x)$ if $K(x) < n$ and $K(x : \overline{n}|) = n$ if $K(x) \geq n$. The above is in fact the number of payments for the discretely payable annuity due.

$$\mathbf{E}[\ddot{a}_{\overline{K(x:\overline{n}|)+1}|}] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|k|} q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{n}|k|} q_x := \ddot{a}_{x:\overline{n}|}.$$

(Recall the endowment insurance case.) Recall, however that the p.m.f. is

$${}_kp_{x:\overline{n}|} \cdot q_{(x:\overline{n}|)+k} = \begin{cases} {}_kp_x \cdot q_{x+k}, & k \leq n-2 \\ {}_{n-1}p_x, & k = n-1 \\ 0, & k \geq n \end{cases}$$

Thus

$$\mathbf{E}[\ddot{a}_{\overline{K(x:\overline{n}|)+1}|}] = \sum^{n-2} \ddot{a}_{\overline{k+1}|k|} q_x + \ddot{a}_{\overline{n}|n-1}p_x$$

Example 0.2 (cont.)

Therefore

$$\begin{aligned}
\mathbf{E}[\ddot{a}_{\overline{K(x:\overline{n}|)+1}|}] &= \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|k|}q_x + \ddot{a}_{\overline{n}|} \sum_{k=n-1}^{\infty} {}_{k|}q_x \\
&= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|k|}q_x + \ddot{a}_{\overline{n}|} \sum_{k=n}^{\infty} {}_{k|}q_x \\
&= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|k|}q_x + \ddot{a}_{\overline{n}|}{}_np_x
\end{aligned}$$

Also

$$\mathbf{E}[\bar{a}_{\overline{T(x:\overline{n}|)}|}] = \int_0^n \bar{a}_{\overline{t}|}{}_tp_x\,\mu(x+t)dt + \bar{a}_{\overline{n}|} \cdot {}_np_x\,.$$

Proposition 0.5

For $(u) = (x : \overline{n|})$, *we have that*

$$\ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} {}_kE_x.$$

Proof.

$$\begin{aligned}
\ddot{a}_{x:\overline{n|}} &= \sum_{k=0}^{n-1}\sum_{i=0}^{k} v^i {}_{k|}q_x + \sum_{i=0}^{n-1} v^i {}_np_x \\
&= \sum_{i=0}^{n-1}\sum_{k=i}^{n-1} v^i {}_{k|}q_x + \sum_{i=0}^{n-1} v^i {}_np_x \\
&= \sum_{i=0}^{n-1} v^i \left(\sum_{k=i}^{n-1} {}_{k|}q_x + {}_np_x\right) = \sum_{i=0}^{n-1} {}_ip_x v^i,
\end{aligned}$$

as required. □

Proposition 0.6

For a general (u) (be careful with the joint life statuses), we have that

$$\ddot{a}_u = -\ddot{a}_{\overline{b+1}|b+1}p_u + \ddot{a}_{\overline{a}|a}p_u + \sum_{k=a}^{b} {}_kp_u v^k.$$

Proof.

Write the annuity as, for a and b being any in $\{0,\ 1, \ldots, \infty\}$.

$$\ddot{a}_u = -\sum_{k=a}^{b} \ddot{a}_{\overline{k+1}|}\Delta_k p_u.$$

$$\begin{aligned}
\ddot{a}_u &= -\left(\ddot{a}_{\overline{k+1}|k}p_u\Big|_a^{b+1} - \sum_{k=a}^{b} {}_{k+1}p_u \Delta\ddot{a}_{\overline{k+1}|}\right) \\
&= -\ddot{a}_{\overline{b+2}|b+1}p_u + \ddot{a}_{\overline{a+1}|a}p_u + \sum_{} {}_{k+1}p_u v^{k+1}.
\end{aligned}$$

Proof.

Further

$$\begin{aligned}
\ddot{a}_u &= -\ddot{a}_{\overline{b+2}|b+1}p_u + \ddot{a}_{\overline{a+1}|a}p_u + \sum_{k=a+1}^{b+1} {}_kp_u v^k \\
&= -\ddot{a}_{\overline{b+1}|b+1}p_u + \ddot{a}_{\overline{a}|a}p_u + \sum_{k=a}^{b} {}_kp_u v^k
\end{aligned}$$

Example 0.3 (Deferred annuities.)

We are sometimes interested in an annuity that starts to pay one dollar after n time units only. We thus have, e.g.,

$$\ddot{a}_u - \ddot{a}_{u:\overline{n}|} := {}_n|\ddot{a}_u = \sum_{k=n}^{\infty} {}_kp_u v^k.$$

Also,

$$\begin{aligned}
{}_n|\ddot{a}_x &= \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|k}|q_x - \ddot{a}_{\overline{n}|} \sum_{k=n}^{\infty} {}_k|q_x = \sum_{k=n}^{\infty} \left(\ddot{a}_{\overline{k+1}|} - \ddot{a}_{\overline{n}|}\right) {}_k|q_x \\
&= \sum_{k=0}^{\infty} \left(\ddot{a}_{\overline{k+n+1}|} - \ddot{a}_{\overline{n}|}\right) {}_{k+n}|q_x = \sum_{k=0}^{\infty} \left(\frac{v^n(1-v^{k+1})}{d}\right) {}_{k+n}|q_x \\
&= v^n \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} {}_{n+k}p_x \cdot q_{x+n+k} = {}_nE_x \cdot \ddot{a}_{x+n}.
\end{aligned}$$

Example (cont.)

For the continuous case, we have that (make sure you can prove that),

$$ {}_{n|}\overline{a}_x = {}_nE_x\overline{a}_{x+n}, $$

as well as

$$ {}_{n|}\overline{a}_x = \int_n^\infty {}_tE_x dt. $$

Note that we can obtain, say the discretely payable deferred whole life annuity as an expectation of the r.v.

$$ K^*(x) := \begin{cases} \ddot{a}_{\overline{K(x)+1|}} - \ddot{a}_{\overline{K(x)+1|}} \equiv 0, & K(x) < n \\ \ddot{a}_{\overline{K(x)+1|}} - \ddot{a}_{\overline{n|}}, & K(x) \geq n \end{cases}. $$

Example 0.4 (n year certain and then life annuity)

Consider an r.v.

$$K^*(x) := \begin{cases} \ddot{a}_{\overline{n}|}, & K(x) < n \\ \ddot{a}_{\overline{K(x)+1}|}, & K(x) \geq n \end{cases}.$$

Then taking expectation, we have that

$$\begin{aligned} \ddot{a}_{x:\overline{n}|} &= \ddot{a}_{\overline{n}|n}q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|k|}q_x = \ddot{a}_{\overline{n}|n}q_x - \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|}\Delta_k p_x \\ &= \ddot{a}_{\overline{n}|n}q_x - \left(\ddot{a}_{\overline{k+1}|k}p_x \Big|_n^{\infty} - \sum_{k=n}^{\infty} {}_{k+1}p_x v^{k+1} \right) \\ &= \ddot{a}_{\overline{n}|} + v^n {}_np_x + \sum_{k=n}^{\infty} {}_{k+1}p_x v^{k+1} \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x. \end{aligned}$$

In a similar fashion $\bar{a}_{x:\overline{n}|} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x = \bar{a}_{\overline{n}|} + {}_nE_x\bar{a}_{x+n}$.

Proposition 0.7

The variance of the n year certain and whole life annuity is equal to the variance of the n years deferred annuity.

Proof.

Note that if we denote by $K^*(x)$ the random present value of the deferred annuity and by $k^{**}(x)$ the random present value of the n year certain whole life annuity, then

$$k^{**}(x) = \ddot{a}_{\overline{n}|} + K^*(x).$$

Thus

$$\mathbf{Var}[k^{**}(x)] = \mathbf{Var}[\ddot{a}_{\overline{n}|}] + \mathbf{Var}[K^*(x)] + \mathbf{Cov}[\ddot{a}_{\overline{n}|}, K^*(x)],$$

as required. □

Recall.

We know that

$$\overline{s}_{\overline{n}|} := (1+i)^n \overline{a}_{\overline{n}|},$$

for $n \geq 0$. The above is an accumulated present value of an annuity at time n instead of time 0. Similarly, we have that

$$\overline{s}_{x:\overline{n}|} := ({}_nE_x)^{-1}\, \overline{a}_{x:\overline{n}|},$$

that represents the actuarial accumulated value at the end of the term of an n-year temporary life annuity of 1 per year payable continuously while (x) survives. The benefit is available at age $x+n$ if x survives till then

Annuity Name	Present-Value Random Variable Y	Actuarial Present Value $E[Y]$ Equal to
Whole Life Annuity		
• Due	$\ddot{a}_{\overline{K+1}\rceil} \quad K \geq 0$	$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$
• Immediate	$a_{\overline{K}\rceil} \quad K \geq 0$	$a_x = \sum_{k=1}^{\infty} v^k \, {}_kp_x$
n-Year Temporary Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{K+1}\rceil} & 0 \leq K < n \\ \ddot{a}_{\overline{n}\rceil} & K \geq n \end{cases}$	$\ddot{a}_{x:\overline{n}\rceil} = \sum_{k=0}^{n-1} v^k \, {}_kp_x$
• Immediate	$\begin{cases} a_{\overline{K}\rceil} & 0 \leq K < n \\ a_{\overline{n}\rceil} & K \geq n \end{cases}$	$a_{x:\overline{n}\rceil} = \sum_{k=1}^{n} v^k \, {}_kp_x$
n-Year Deferred Whole Life Annuity		
• Due	$\begin{cases} 0 & 0 \leq K < n \\ \ddot{a}_{\overline{K+1}\rceil} - \ddot{a}_{\overline{n}\rceil} & K \geq n \end{cases}$	${}_{n\rceil}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \, {}_kp_x$
• Immediate	$\begin{cases} 0 & 0 \leq K < n \\ a_{\overline{K}\rceil} - a_{\overline{n}\rceil} & K \geq n \end{cases}$	${}_{n\rceil}a_x = \sum_{k=n+1}^{\infty} v^k \, {}_kp_x$
n-Year Certain and Whole Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{n}\rceil} & 0 \leq K < n \\ \ddot{a}_{\overline{K+1}\rceil} & K \geq n \end{cases}$	$\ddot{a}_{\overline{x:\overline{n}\rceil}} = \ddot{a}_{\overline{n}\rceil} + \sum_{k=n}^{\infty} v^k \, {}_kp_x$
• Immediate	$\begin{cases} a_{\overline{n}\rceil} & 0 \leq K < n \\ a_{\overline{K}\rceil} & K \geq n \end{cases}$	$a_{\overline{x:\overline{n}\rceil}} = a_{\overline{n}\rceil} + \sum_{k=n+1}^{\infty} v^k \, {}_kp_x$

Summary of discrete life annuities of 1 per annum payable at the beginning of each year (due) or at the end of each year (immediate)

Additional relations are

- $1 = d\,\ddot{a}_x + A_x$
- $A_x = v\,\ddot{a}_x - a_x$
- $1 = d\,\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}$
- $\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$
- $A^1_{x:\overline{n}|} = v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$
- $A_{x:\overline{n}|} = v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$
- ${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$
- $\ddot{a}_{\overline{x:n}|} = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$
- $\ddot{s}_{x:\overline{n}|} = \dfrac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x}$

$$= \sum_{k=0}^{n-1} (1+i)^{n-k} \frac{l_{x+k}}{l_{x+n}}.$$

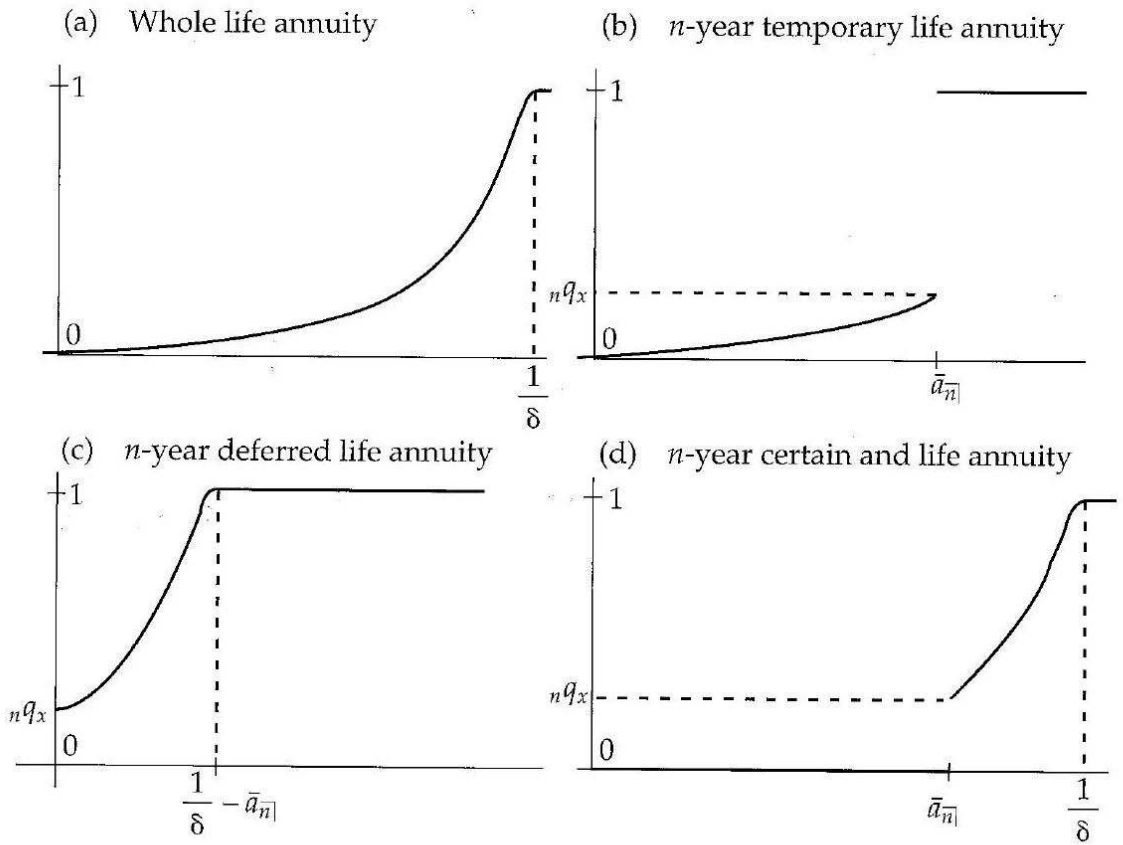
Useful relations between specific discrete annuities and insurances.

Annuity Name	Present-Value Random Variable Y	Actuarial Present Value $E[Y]$ Equal to
Whole Life Annuity	$\bar{a}_{\overline{T}\vert} \quad T \geq 0$	$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$
n-Year Temporary Life Annuity	$\begin{cases} \bar{a}_{\overline{T}\vert} & 0 \leq T < n \\ \bar{a}_{\overline{n}\vert} & T \geq n \end{cases}$	$\bar{a}_{x:\overline{n}\vert} = \int_0^n v^t \, {}_tp_x \, dt$
n-Year Deferred Whole Life Annuity	$\begin{cases} 0 & 0 \leq T < n \\ \bar{a}_{\overline{T}\vert} - \bar{a}_{\overline{n}\vert} & T \geq n \end{cases}$	${}_{n\vert}\bar{a}_x = \int_n^\infty v^t \, {}_tp_x \, dt$
n-Year Certain and Life Annuity	$\begin{cases} \bar{a}_{\overline{n}\vert} & 0 \leq T < n \\ \bar{a}_{\overline{T}\vert} & T \geq n \end{cases}$	$\bar{a}_{\overline{x:n}\vert} = \bar{a}_{\overline{n}\vert} + \int_n^\infty v^t \, {}_tp_x \, dt$

Additional relations are

- $1 = \delta \, \bar{a}_x + \bar{A}_x$
- $1 = \delta \, \bar{a}_{x:\overline{n}\vert} + \bar{A}_{x:\overline{n}\vert}$
- ${}_{n\vert}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}\vert}$
- $\bar{s}_{x:\overline{n}\vert} = \dfrac{\bar{a}_{x:\overline{n}\vert}}{{}_nE_x} = \int_0^n (1+i)^{n-t} \dfrac{l_{x+t}}{l_{x+n}} \, dt.$

Summary of continuous life annuities of 1 per annum payable continuously.



Typical distributions for the present value rv T.